4. The alternating groups

(E4.1) Every element of $\mathrm{Sym}(\Omega)$ can be written as a finite product of transpositions.[2]

(E4.2) If Ω is an infinite set, then one defines the *finitary symmetric group* to be the set of all permtuations that fix all but a finite number of elements of Ω.

1. Prove that $\mathrm{FinSym}(\Omega)$ is a group.
2. Prove that $\mathrm{FinSym}(\Omega)$ is generated by the set of all transpositions.
3. Prove that the function sgn given at (3) is a group homomorphism from $\mathrm{FinSym}(\Omega)$ to C_2.
4. (Harder) Prove that the kernel of sgn (known as the *finitary alternating group*) is an infinite simple group.

(E4.3) Let g, h be two elements of $\mathrm{Sym}(\Omega)$. Then g and h are conjugate in $\mathrm{Sym}(\Omega)$ if and only if they have the same cycle type.

(E4.4) Let C be a conjugacy class of $\mathrm{Sym}(\Omega)$ corresponding to partition $1^{n_1}2^{n_2}3^{n_3}\cdots$. Then $C \subset \mathrm{Alt}(\Omega)$ if and only if

$$n_2 + n_4 + n_6 + \cdots$$

is even.

(E4.5) Let C be a conjugacy class of $\mathrm{Sym}(n)$ of type $1^{a_1}2^{a_2}3^{a_3}\cdots$. Suppose that $g \in C \subset \mathrm{Alt}(n)$. The following are equivalent:

1. C is the union of two conjugacy class of $\mathrm{Alt}(n)$;
2. $a_i \leq 1$ for all i, with $a_i = 0$ for all even i.

(E4.6) Prove that, if $n \geq 5$ and C is a non-trivial conjugacy class of $\mathrm{Alt}(n)$, then $|C| > n$.

(E4.7) The set C is the union of a number of conjugacy classes, $C_1, \dots, C_k$, of N; the classes $C_1, \dots C_k$ are of equal size; finally k divides $|G : N|$.

(E4.8) Write down the subgroup lattice of $\mathrm{Alt}(4)$. Identity which subgroups are normal and thereby demonstrate that $\mathrm{Alt}(4)$ is not simple. Prove that $\mathrm{Alt}(2)$ and $\mathrm{Alt}(3)$ are simple and abelian.

(E4.9) Prove that the group $\mathrm{Alt}(n)$ is generated by the set of all 3-cycles (a *3-cycle* is an element of cycle type $1^{n-3}3^1$). Show, in fact, that the following set of 3-cycles is sufficient to generate $\mathrm{Alt}(n)$:

$$\{(1,2,i) \mid i = 3, \dots, n\}.$$

(E4.10) Suppose that the action of H on K is the trivial action. What is $K \rtimes_\phi H$?

(E4.11) Suppose that K is a normal subgroup of a group G with G/K isomorphic to a group H. The extension $H.K$ is split if and only if G contains a subgroup J such that $G = JK$ and $J \cap K = \{1\}$.

(E4.12) Prove that, for all integers $n \geq 2$, $\mathrm{Sym}(n) \cong \mathrm{Alt}(n) : C_2$.

(E4.13) Find an example of a group $G = K.H$ (where K and H are both non-trivial finite groups) which is non-split. Hint: there is precisely one example with $|G| \leq 7$, and it is abelian. The smallest non-abelian examples have $|G| = 8$.

(E4.14) Write down as many groups G as you can, for which $G = K.H$ where $K \cong A_6$ and $H \cong C_2$. Identify those that can be written as split extensions.

(E4.15) Prove that if $H \leq N_G(K)$, then $HK = KH$, and HK is a group.

(E4.16) Prove that a group G is almost simple if and only if the following two conditions hold:

1. G contains a normal subgroup S that is non-abelian and simple;
2. any non-trivial normal subgroup of G contains S.

(E4.17) Prove that $\mathrm{Sym}(n)$ is almost simple for $n \geq 5$.

(E4.18) (Hard) How many almost simple groups (up to isomorphism) have a normal subgroup isomorphic to $\mathrm{Alt}(6)$?

(E4.19) If $n \geq 3$ and $n \neq 6$, then any automorphism of $\mathrm{Sym}(n)$ is inner. Thus $\mathrm{Aut}(\mathrm{Sym}(n)) = \mathrm{Sym}(n)$.

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[2]Put another way - and using terminology introduced in the previous chapter - this exercise asserts, precisely, that $\mathrm{Sym}(\Omega)$ is *generated* by the set of all transpositions.

(E4.20) Let ϕ be an automorphism of a group G and let $g, h \in G$. Then

- g and h have the same order;
- $C_G(g) \cong C_G(\phi(g))$;
- If g and h are conjugate in G, then $\phi(g)$ and $\phi(h)$ are conjugate in G.[3]

(E4.21) Suppose that H is a subgroup of a group G and suppose that, for all $g \in G$, $g^2 \in H$. Then $|G : H| \leq 2$. Is this result true for integers other than 2?

(E4.22) Prove that Alt(5) contains 6 Sylow 5-subgroups.

(E4.23) Prove that, in fact, $H \hookrightarrow \text{Alt}(6)$. Prove, moreover, that H has 6 distinct conjugates in Alt(6).

(E4.24) Prove that this isomorphism is not induced by an element of Sym(6).

(E4.25) Let Ω be a finite set of order n, and let Γ be a subset of Ω with $|\Gamma| = k$.

(1) There is a unique subgroup G of $\text{Sym}(\Omega)$ that preserves Γ setwise and is isomorphic to $\text{Sym}(k) \times \text{Sym}(n-k)$;
(2) if $H \leq \text{Sym}(\Omega)$ preserves Γ setwise, then $H \leq G$.

(E4.26) Consider a category called **Intrans**

Objects: An object is a pair (Γ, Δ) where Γ is a finite set and Δ is a subset of Γ.

Arrows: An arrow $(\Gamma, \Delta) \to (\Gamma', \Delta')$ is a function $f : \Gamma \to \Gamma'$ such that $x \in \Delta \implies f(x) \in \Delta'$.

(1) Prove that **Intrans** is a category.
(2) Prove that if X is an object in **Intrans**, then $\text{Aut}(X) \cong \text{Sym}(\Delta) \times \text{Sym}(\Gamma \backslash \Delta)$.
(3) Prove that if G acts on $X = (\Gamma, \Delta)$ as an object from **Intrans**, then G is a subset of the setwise stabilizer of Δ, and conversely.

(E4.27) Let Ω be a subset of order n and let Γ and Δ) be subsets of Ω of order k. Let H (resp. K) be the setwise stabilizer of Γ (resp. Δ) in $\text{Sym}(n)$. For what values of n and k is H maximal? Are H and K conjugate? How many conjugacy classes of subgroups isomorphic to H does $\text{Sym}(n)$ contain?

(E4.28) Describe the intersection of $\text{Sym}(k) \times \text{Sym}(n-k)$ with $\text{Alt}(n)$. Is it maximal in $\text{Alt}(n)$? How many conjugacy classes of such subgroups are there?

[3] In particular this implies that $\text{Aut}(G)$ has a well-defined action on the set of conjugacy classes of G. This is another way of looking at the situation described in §**??**.