3. Group actions

Throughout this section G is a group and Ω is a set. A *(right) action* of G on Ω is a function

$$\varphi : G \times \Omega \to \Omega,\ (g, \omega) \mapsto \omega^g \tag{1}$$

such that

(A1) $\omega^1 = \omega$ for all $\omega \in \Omega$;
(A2) $(\omega^g)^h = \omega^{gh}$ for all $\omega \in \Omega$ and $g, h \in G$.

We will refer to the triple (G, Ω, φ) as a *G-set*, since it is an object in the category **G-Set**. Let us briefly discuss some examples, the first is particularly fundamental.

Example 10. Let Ω be a set and let G be any subgroup of $\mathrm{Sym}(\Omega)$, the symmetric group on Ω. The group G acts naturally on Ω via the action (1) where we write ω^g to mean the image of the element ω under the permutation g.

(E3.1) *Verify that the function described in Example 10 is an action, i.e. that (A1) and (A2) hold.*

Example 11. Let V be a vector space and let G be any subgroup of $\mathrm{GL}(V)$, the general linear group on V. The group G acts naturally on V via

$$G \times V \to V,\ (g, v) \mapsto v \cdot g$$

Here we write $v \cdot g$ to mean application (on the right) of the linear transformation g to the vector v. If V is finite-dimensional, then we can take a basis and write v as a row vector, g as a square matrix, and $v \cdot g$ becomes just matrix multiplication.

(E3.2) *Verify that the function described in Exercise 11 is an action.*

(E3.3) *Suppose that we changed the function described in Exercise 11 from $v \cdot g$ to $g \cdot v$ (so, for instance, if V is finite-dimensional we consider v as a column vector and use matrix multiplication). Show that this is not an action. Can you find a 'natural' adjustment to this definition so that it becomes an action?*

The next example is a specific instance of Example 10. To describe it we need a little bit of notation. Suppose that A is a subset of a group G. Define

$$\langle A \rangle := \{a_1 a_2 \cdots a_k \mid k \in \mathbb{Z}^+, a_i \in A \text{ or } a_i^{-1} \in A\}.$$

It should be clear that $\langle A \rangle$ is a group. In fact $\langle A \rangle$ is the smallest subgroup of G containing A and we refer to it as *the group generated by A.*

(E3.4) *Give sufficient conditions such that*

$$\langle A \rangle = \{a_1 a_2 \cdots a_k \mid k \in \mathbb{Z}^+, a_i \in A\}.$$

Give an example of a set A in a group G for which this inequality does not hold.

Example 12. Let $\Omega = \{1, \ldots, n\}$ with $3 \le n \in \mathbb{Z}^+$. Consider the group $G = \langle g, h \rangle \le \mathrm{Sym}(\Omega)$ where

$$g = (1, 2, \ldots, n) \text{ and } h = (1, n-1)(2, n-2) \ldots \left(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\right)$$

The group G is known as D_{2n}, the dihedral group of order $2n$ and in the ensuing exercise we will establish some standard facts about this group.

Let us make an observation about the action of G on the set Ω that will become relevant shortly. We can think of the set Ω as the set of vertices of an object $X = (\Omega, E)$ from the category **SimpleGraph**, where E is the set

$$\{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}.$$

Clearly X can be represented by drawing a regular n-gon and labelling the vertices, in order anti-clockwise, $1, \ldots, n$; see Figure 1 for an example when $n = 5$. Notice that

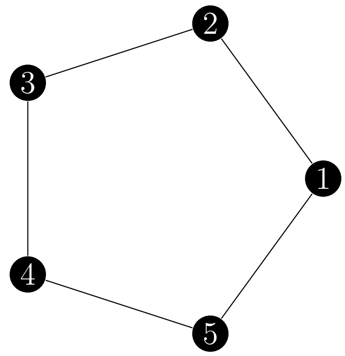

FIGURE 1. D_{10} acts on the pentagon with $g = (1, 2, 3, 4, 5)$ and $h = (2, 5)(3, 4)$.

the permutation g simply rotates the pentagon anti-clockwise by an angle of $\frac{2\pi}{n}$, while the permutation h reflects the polygon through a line passing through the centre and the vertex marked n. In particular, if $\{i, j\} \in E$, then

$$\{i^g, j^g\} \in E \text{ and } \{i^h, j^h\} \in E.$$

(E3.5) *Prove that the order of g (resp. h) is n (resp. 2), and that $h^{-1}gh = g^{-1}$. Prove that G is of order $2n$ and that G contains a normal cyclic subgroup C of order n. Prove that every element in $G\backslash C$ has order 2.*

3.1. **Actions and morphisms into the symmetric group.** The following lemma is, hopefully, known to you.

Lemma 3.1. *Let φ be an action of a group G on a set Ω. For $g \in G$, define the function $\varphi_g^* : \Omega \to \Omega, \omega \to \omega^g$. Then the function*

$$\varphi^* : G \to \mathrm{Sym}(\Omega), g \to \varphi_g^*$$

is a group homomorphism. Conversely, given a group homomorphism $\theta : G \to \mathrm{Sym}(\Omega)$, then the function

$$\theta^\dagger : G \times \Omega \to \Omega, (g, \omega) \mapsto \omega^{\theta(g)}$$

is an action. Moreover $(\varphi^*)^\dagger = \varphi$ and $(\theta^\dagger)^* = \theta$.

Proof. Assume that φ is an action. The axioms for an action imply that φ_g^* has inverse equal to $\varphi_{g^{-1}}^*$ hence, in particular, φ_g^* is a bijection and so $g \to \varphi_g^*$ is a well-defined function with codomain $\mathrm{Sym}(\Omega)$. Now

$$\begin{aligned} \varphi^*(gh) = \varphi_{gh}^* &= (\omega \mapsto \omega^{gh}) \\ &= (\omega \mapsto (\omega^g)^h) \\ &= \varphi^*(g) \cdot \varphi^*(h) \end{aligned}$$

and so $g \to \varphi_g^*$ is a group homomorphism as required.

Conversely, given θ, we know that $\theta(1) = 1 \in \mathrm{Sym}(G)$ and so $\omega^\theta(1) = \omega$ as required. Similarly

$$\theta^\dagger(gh, \omega) = \omega^{\theta(gh)} = \omega^{\theta(g)\theta(h)} = (\omega^{\theta(g)})^{\theta(h)}$$

and the second implication holds.

The final statement ('Moreover...') is left for the reader. $\square$

This lemma amounts to the equivalence of the two statements 'G acts on Ω' and 'there is a homomorphism $G \to \mathrm{Sym}(\Omega)$.' Observe that this is simply a generalization of Example 10 in which we discussed the natural action of a subgroup G in $\mathrm{Sym}(\Omega)$ acting on the set Ω. Recall that G can be thought of as a subgroup of $\mathrm{Sym}(\Omega)$ whenever there is an injective homomorphism from G to $\mathrm{Sym}(\Omega)$

- the lemma asserts that the example still holds even without 'injectivity' and that, effectively, all actions take this form. [7]

In what follows when we are given an action $\varphi : G \times \Omega \to \Omega$ we will refer to $\varphi^* : G \to \mathrm{Sym}(\Omega)$ as the *associated homomorphism*.

3.1.1. *Other categories.* Our work in §2 suggests a generalization of Lemma 3.1. First we need some terminology for which we refer to the function φ_* defined in Lemma 3.1.

Let X be an object in a category $\mathbf{C}$ of structured sets. We say that φ is an action of a group G *on X as an object in* $\mathbf{C}$ if φ is an action on the underlying set Ω such that, for every $g \in G$, the map φ_g^* is an arrow in $\mathbf{C}$.

> **(E3.6)** *Check that the action described in Example 12 is a well-defined action of G on X as an object from* **SimpleGraph**.

Lemma 3.2. *Let X be an object in a category $\mathbf{C}$ of structured sets. Let φ be an action of a group G on a X as an object in $\mathbf{C}$. Then the function*

$$\varphi^* : G \to \mathrm{Aut}(X), g \to \varphi_g^*$$

is a group homomorphism. Conversely, given a group homomorphism $\theta : G \to \mathrm{Aut}(X)$, then the function

$$\theta^\dagger : G \times X \to X, (g, \omega) \mapsto \omega^{\theta(g)}$$

is an action of G on X as an object in $\mathbf{C}$. Moreover $(\varphi^*)^\dagger = \varphi$ and $(\theta^\dagger)^* = \theta$.

Proof. Since, by assumption, the map φ_g^* is an arrow in $\mathbf{C}$ for every g, one immediately obtains that φ_g^* and $\varphi_{g^{-1}}^*$ are an inverse form of isomorphisms. In particular the map φ^* is a well-defined function into the set $\mathrm{Aut}(X)$. The rest of the proof is now (virtually) word for the word the same as the previous. $\square$

> **(E3.7)** *See if you can define analogues of the categories* $\mathbf{G-Set}$ *and* $\mathbf{G-Set2}$ *for which Lemma 3.2 amounts to a statement about equivalence of categories.*

The most important example relating to Lemma 3.2 is for the situation $\mathbf{C} = \mathbf{Set}$, which is the case covered by Lemma 3.1. Let us mention one other example (another will crop up in the next subsection).

> **Example 13.** Let $\mathbf{C} = \mathbf{Vect}_K$ and let V be an object in $\mathbf{C}$, i.e. V is a vector space over the field K. A group G acts on V as an object in $\mathbf{C}$ if, for all $g \in G$, the map $v \mapsto v^g$ is a linear transformation of V, i.e. if
>
> $$(c \cdot v + d \cdot w)^g = c \cdot v^g + d \cdot w^g, \text{ for all } c, d \in K; v, w \in V.$$
>
> Lemma 3.2 asserts that prescribing such an action is equivalent to prescribing a group homomorphism $\theta : G \to \mathrm{GL}(V)$, the general linear group of V. There is a whole field of mathematics dedicated to the study of such morphisms, namely the field of *representation theory*.
>
> Note that the action of G on V as an object in $\mathbf{Vect}_K$ is still an action of G on V thought of as an object in $\mathbf{Set}$.[8] Thus prescribing an action of G on V as an object in $\mathbf{Vect}_K$ by default yields a group homomorphism $\theta : G \to \mathrm{Sym}(V)$.
>
> By observing that the action preserves the structure of a vector space, though, we obtain a lot of information about the location of the image of θ in $\mathrm{Sym}(V)$. In particular, the group $\mathrm{GL}(V)$ is a proper subgroup of $\mathrm{Sym}(V)$ and so the fact that

[7] Here is yet another point of view: define a new category $\mathbf{G-Set2}$ whose objects are triples (G, Ω, θ) where $\theta : G \to \mathrm{Sym}(\Omega)$ is a group homomorphism. Now the lemma asserts that $\mathbf{G-Set}$ and $\mathbf{G-Set2}$ are *equivalent categories*.

[8] To be rigorous, I should apply the obvious forgetful functor $\mathbf{Vect} \to \mathbf{Set}$ here.

the action preserves the structure of a vector space is equivalent to knowing that the image of θ is a subgroup of this proper subgroup.

3.2. **Properties of actions.** Let (G, Ω, φ) be a G-set. Define,

- for $\omega \in \Omega$, $G_\omega := \{g \in G \mid \omega^g = \omega\}$, is the *stabilizer* of ω;
- $G_{(\Omega)} := \bigcap_{\omega \in \Omega} G_\omega$ is the *kernel* of the action;
- for $\omega \in \Omega$, $\omega^G := \{\omega^g \mid g \in G\}$ is the *orbit* of ω.

(E3.8) *Show that G_ω is a subgroup of G for all $\omega \in \Omega$. Show that $G_{(\Omega)}$ is a normal subgroup of G, equal to the kernel of the associated homomorphism ϕ^*.*

(E3.9) *Suppose that a group G acts on a set Ω. Show that the set of orbits*

$$\{\omega^G \mid \omega \in \Omega\}$$

partitions Ω.

We say that the action of G on Ω is

- *faithful*, if $G_{(\Omega)} = \{1\}$; equivalently, the associated homomorphism φ^* is a monomorphism and we think of G as a subgroup of $\mathrm{Sym}(\Omega)$;
- *transitive*, if $\omega^G = \Omega$ for some (and hence all) $\omega \in \Omega$.
- *semiregular*, if $G_\omega = \{1\}$ for all $\omega \in \Omega$;[9]
- *regular*, if the action is transitive and semiregular.

(E3.10) *let $G = \mathrm{Sym}(\Omega)$ in Example (E1.1). Prove that the action is faithful. Under what conditions is it transitive (resp. semiregular)? Describe the stabilizer of an element of Ω.*

(It may be easier to restrict to the case where Ω is finite. In which case we can choose a labelling so that $\Omega = \{1, \dots, n\}$, for a positive integer n.)

(E3.11) *let $G = \mathrm{GL}(V)$ in Example (E1.1). Prove that the action is faithful. Under what conditions is it transitive (resp. semiregular)? Describe the stabilizer of the zero vector. Let V be finite-dimensional, choose a basis $\{e_1, \dots, e_n\}$ and describe the stabilizer of e_1.*

(E3.12) *Consider the action described in Example 12. Prove that the action is both faithful and transitive (and hence the action induces an embedding of D_{2n} in $\mathrm{Aut}(X)$). What are the vertex-stabilizers in this action? When does $D_{2n} = \mathrm{Aut}(X)$?*

3.2.1. *Permutation groups.* Let $\varphi : G \times \Omega \to \Omega$ be an action, and let $\theta : G \to \mathrm{Sym}(\Omega)$ be the associated homomorphism. It is clear that the kernel, $G_{(\Omega)}$, of φ, is equal to the kernel of θ as a group homomorphism. In particular, this means that φ is faithful if and only if θ is injective, i.e. θ is an embedding of G in $\mathrm{Sym}(\Omega)$.

In the literature, a *permutation group* is an abstract group G accompanied by some fixed embedding of G in $\mathrm{Sym}(\Omega)$, for some set Ω. Equivalently, a permutation group is an abstract group G accompanied by some faithful action.

3.3. **Actions from another point of view.** We need to know when two actions are 'the same'. Hopefully our work in the previous chapter demonstrates that this notion is provided by the concept of an an isomorphism in the category **G-set**. Such an isomorphism is known in the literature as a permutation isomorphism and we now define it explicitly.

A *permutation isomorphism* between two G-sets (G, Ω, φ) and (H, Γ, ψ) is a pair $\alpha : G \to H, \beta : \Omega \to \Gamma$, where α is a group isomorphism, β is a bijection, and the following diagram commutes:

[9] In other disciplines, notably algebraic topology and geometric group theory, people tend not to use the term 'semiregular', and say instead that 'G acts freely on Ω.'

$$\begin{tikzcd} G \times \Omega \arrow[r, "\varphi"] \arrow[d, "(\alpha,\beta)"'] & \Omega \arrow[d, "\beta"] \\ H \times \Gamma \arrow[r, "\psi"] & \Gamma \end{tikzcd} \tag{2}$$

In what follows we will use terms like 'two actions are isomorphic' to mean that the associated G-sets are isomorphic.

The next example, and ensuing lemma, shows that any *transitive* action has a particular form.

> **Example 14.** Let H be any subgroup of G. The group G acts transitively on $H\backslash G$, the set of right cosets of H via $(Ha, g) \mapsto Hag$ (i.e. by right multiplication). When H is trivial, this is called the *right regular action* of G.
>
> Similarly G acts transitively on G/H, the set of left cosets of H via $(aH, g) \mapsto g^{-1}aH$. When H is trivial, this is called the *left regular action* of G.

Lemma 3.3. *Suppose that a group G acts transitively on a set Ω, and let H be the stabilizer in G of some point ω of Ω. Then the action of G on Ω is isomorphic to the action of G on $H\backslash G$.*

Proof. Let $\alpha : G \to G$ be the identity map. Let γ be an element of Ω. Since G is transitive, there exists $g \in G$ such that $\omega^g = \gamma$. Observe that if f is an element of the coset Hg, then $\omega^f = \gamma$.

Conversely suppose that $k \in G$ satisfies $\omega^k = \gamma$. Then

$$\omega^{kg^{-1}} = (\omega^k)^{g^{-1}} = \gamma^{g^{-1}} = \omega$$

and so $kg^{-1} \in H$, the stabilizer of ω. We conclude that $k \in Hg$ and so Hg is precisely the set of all elements f in G for which $\omega^f = \gamma$.

Now define

$$\beta : \Omega \to H\backslash G, \ \gamma \mapsto Hg$$

where g is an element that maps ω to γ. The work of the previous paragraph implies that this definition is well-defined. Now one just needs to verify that (2) holds and we are done. $\square$

> **(E3.13)** *Verify that (2) holds, thereby completing the proof of Lemma 3.3.*
>
> **(E3.14)** *What conditions on H are equivalent to the action of G on $H\backslash G$ being faithful?*

When the group G is finite, we can apply Lemma 3.3 to obtain the following important result, which is known as the *Orbit-Stabilizer Theorem.*

Theorem 3.4. *Suppose that a finite group G acts on a set Ω. Then, for all $\omega \in \Omega$,*

$$|G| = |G_\omega| \cdot |\omega^G|.$$

Proof. Write $\Gamma := \omega^G$. Clearly G acts naturally on Γ and, by definition, this action is transitive. Thus Lemma 3.3 applies and the action of G on Γ is equivalent to the action of G on G_ω/G. In particular,

$$|\Gamma| = |G : G_\omega| = \frac{|G|}{|G_\omega|}$$

and we are done. $\square$

The Orbit-Stabilizer Theorem has many obvious consequences for the action of a finite group G. For instance, using Lagrange's theorem, we see that the order of any orbit divides $|G|$. If, in particular, the action is semi-regular, then the length of any orbit is equal to $|G|$ (indeed, the converse is also obviously true).

(E3.15) *Let G be a finite group acting transitively on a set Ω. Show that the average number of fixed points of the elements of G is 1, i.e.*

$$\frac{1}{|G|}\sum_{g\in G}|\{\omega\in\Omega \mid \omega^g=\omega\}|=1.$$

3.4. **Groups acting on groups.** Consider the category **Group**, in which objects are groups and arrows are group homomorphisms. By definition, then, an automorphism in this category is a bijection $\phi : K \to K$, where K is a group, and such that for all $g, h \in K$,

$$\phi(g)\cdot\phi(h)=g\cdot h.$$

The set of all such bijections is the group $\mathrm{Aut}(K)$.

Now we can consider the situation where a group H acts on K as an object from **Group**; this is equivalent (by Lemma 3.2) to the existence of a group homomorphism $\phi : H \to \mathrm{Aut}(K)$.

3.4.1. *Groups acting 'on themselves'.* Any group G acts on itself naturally *by conjugation.* Formally, this action is

$$G\times G\to G,\ (g,h)\mapsto g^{-1}hg.$$

Thus, by Lemma 3.2, this is equivalent to the existence of a homomorphism

$$\phi : G\to \mathrm{Aut}(G), g\mapsto \phi_g$$
$$\text{where } \phi_g : G\to G, h\mapsto g^{-1}hg.$$

(E3.16) *Prove that the map ϕ is a well-defined group homormophism from G to $\mathrm{Aut}(G)$ (and, hence, the action of G on itself by conjugation is an action on itself as an object from* **Group***.)*

An orbit in the conjugacy action is a *conjugacy class* of G, and the stabilizer G_h of an element h is $C_G(h)$, the *centralizer* of h. Note that $\{1\}$ is always a conjugacy class of G, which we call *the trivial conjugacy class.* Note, that by (E3.9), the conjugacy classes partition the group G. If two elements $g, h \in G$ lie in the same conjugacy class, then we say that g and h *are conjugate*; conjugate elements have identical group-theoretic properties.

(E3.17) *Prove that if g and h are conjugate elements of G, then they have the same order.*

(E3.18) *Prove that a normal subgroup of G is a union of conjugacy classes of G.*

(E3.19) *Let N be a normal subgroup of G. Prove that G acts (by conjugation) on N as an object grom* **Group***. (In particular, whenever N is a normal subgroup of G, the conjugation action induces a morphism $G \to \mathrm{Aut}(N)$.*

We define $\mathrm{Inn}(G) := \mathrm{Im}(\phi)$ and call $\mathrm{Inn}(G)$ the *inner automorphism group of G.* The quotient $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the *outer automorphism group of G.*[10] To see that $\mathrm{Out}(G)$ is, indeed, a group, we require the following result.

Lemma 3.5.

(1) $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$;
(2) $\ker(\phi) = Z(G)$.

Proof. **(E3.20)** *Prove this.* ∎

[10]Note that elements of $\mathrm{Out}(G)$ **are not** automorphisms of G – they are cosets of $\mathrm{Inn}(G)$.

There is another 'natural' action of a group G that is connnected to its action by conjugacy on itself. In this action we let Ω be the set of all subgroups of G. Then G acts on Ω by conjugation via

$$G \times \Omega \to \Omega \ (g, H) \mapsto g^{-1}Hg.$$

An orbit in this action is *a conjugacy class of subgroups* of G, and the stabilizer G_H of an element $H \in \Omega$ is $N_G(H)$, the *normalizer* of H. If two subgroups $H, K \leq G$ lie in the same conjugacy class of subgroups, then we say that H and K *are conjugate*.

> **(E3.21)** *Consider the action of G by conjugation on the set of all subgroups of G. If H is a subgroup of G and $\{H\}$ is an orbit under this action, then what type of subgroup is H?*
>
> **(E3.22)** *Prove that if G acts transitively on Ω and G_ω is a stabilizer, then the set of all stabilizers equals the set of all conjugates of G_ω.[11] Under what conditions is the action of G by conjugation on this set of conjugates permutation isomorphic to the action of G on Ω?*

3.5. More on permutation isomorphisms.

Lemma 3.6. *If G and H are both permutation groups on Ω, then G and H are permutation isomorphic if and only if G and H are conjugate in* $\mathrm{Sym}(\Omega)$.

Proof. Suppose that G and H are permutation isomorphic. Then there exists a bijection $\beta : \Omega \to \Omega$ and an isomorphism $\alpha : G \to H$ with $(\omega^g)\beta = (\omega\beta)(g\alpha)$ for all $\omega \in \Omega$ and $g \in G$. Applying β^{-1} to both sides, we obtain that

$$\omega^g = ((\omega\beta)^{(g\alpha)})\beta^{-1}$$

and so $g = \beta(g\alpha)\beta^{-1}$ for all $g \in G$. Thus $G = \beta(G\alpha)\beta^{-1} = \beta H \beta^{-1}$ and we are done, since $\beta \in \mathrm{Sym}(\Omega)$.

Conversely, suppose that $G = \beta H \beta^{-1}$ for some $\beta \in \mathrm{Sym}(\Omega)$. Define an isomorphism

$$\psi : G \to H, g \mapsto \beta^{-1} g \beta.$$

Then for all $g \in G$ and $\omega \in \Omega$,

$$\omega^g = \omega^{\beta(\beta^{-1}g\beta)\beta^{-1}} = \omega^{(\beta(g\psi)\beta^{-1}}$$

and so $(\omega^g)\beta = (\omega\beta)^{g\alpha}$ as required. □

The following result is easy, but turns out to be crucial when we come to studying the subgroups of $\mathrm{Sym}(\Omega)$.

Lemma 3.7. *Let G be a permutation group on Ω.*

1. *If $C_{\mathrm{Sym}(\Omega)}(G)$ is transitive on Ω, then G is semiregular.*
2. *If G is transitive on Ω, then $C_{\mathrm{Sym}(\Omega)}(G)$ is semiregular.*

Proof. (1) Let $\alpha, \beta \in \Omega$ and $g \in G_\alpha$. Since $C_{\mathrm{Sym}(\Omega)}(G)$ is transitive on Ω, there exists $h \in C_{\mathrm{Sym}(\Omega)}(G)$ such that $\beta = \alpha^h$. Then

$$\beta^g = \alpha^{hg} = \alpha^{(gh)} = (\alpha^g)^h = \alpha^h = \beta$$

Since β was arbitrary we conclude that g fixes every point of Ω. Thus $g = 1$ and so $G_\alpha = \{1\}$ as required.

(2) Clearly $G \leq C_{\mathrm{Sym}(\Omega)}(C_{\mathrm{Sym}(\Omega)}(G))$ and, since G is transitive, we conclude that $C_{\mathrm{Sym}(\Omega)}(C_{\mathrm{Sym}(\Omega)}(G))$ is transitive also. Thus $C_{\mathrm{Sym}(\Omega)}(G)$ is semiregular by (1). □

> **(E3.23)** *If G is a regular permutation group on Ω then $C_{\mathrm{Sym}(\Omega)}(G)$ is regular.*
>
> **(E3.24)** *If G is a regular permutation group on Ω, then G is permutation isomorphic to $C_{\mathrm{Sym}(\Omega)}(G)$.*

---

[11] In particular if $g, h \in G$ are conjugate, then so are their centralizers.