Answers to the Exercises in Section 3

(E3.23) If G is a regular permutation group on Ω then $C_{\mathrm{Sym}(\Omega)}(G)$ is regular.

> **Answer.** This result will follow from (E3.24) below.

(E3.24) If G is a regular permutation group on Ω, then G is permutation isomorphic to $C_{\mathrm{Sym}(\Omega)}(G)$.

> **Answer.** By Lemma 3.6 we are required to show that G and $C_{\mathrm{Sym}(\Omega)}(G)$ are conjugate subgroups of $\mathrm{Sym}(\Omega)$.
>
> G acting regularly on Ω equates to the action being transitive with trivial stabilizers. We know that the action is, therefore, isomorphic to the action of G on $H\backslash G$ where $H = \{1\}$. Since a coset of H is a singleton, the action of G on $H\backslash G$ is isomorphic to the action of G on itself by right multiplication. In other words we regard the action of G on Ω as being defined by
>
> $$\rho : G \times \Omega \to \Omega, (g, h) \mapsto hg,$$
>
> where $\Omega = G$.[1] The associated homomorphism $\rho^* : G \to \mathrm{Sym}(\Omega)$ yields the original embedding of G in $\mathrm{Sym}(\Omega)$, in other words the group $R = \rho^*(G)$ is G itself.
>
> Let us consider a related action,
>
> $$\lambda : G \times \Omega \to \Omega, (g, h) \mapsto g^{-1}h$$
>
> where, again, $\Omega = G$.[2] Let $\lambda^* : G \to \mathrm{Sym}(\Omega)$ be the associated homomorphism and write $L = \lambda^*(G)$. Since λ is clearly faithful, λ^* is injective, and the first isomorphism theorem implies that $L \cong G$. It is also quite clear that L acts regularly on Ω.
>
> Let us show that L centralizes $R = G$. Take $g, h, x \in G$ and write $\lambda_g = \lambda^*(g) \in L$, $\rho_h = \rho^*(h) \in R$. Then
>
> $$x^{\lambda_g \rho_h} = (g^{-1}x)^{\rho_h} = g^{-1}xh = g^{-1}(x^{\rho_h}) = x^{\rho_h \lambda_g}.$$
>
> Thus $L \leq C_{\mathrm{Sym}(\Omega)}(G)$. On the other hand, Lemma 3.7(ii) implies that $C_{\mathrm{Sym}(\Omega)}(G)$ is semiregular and so, since L is regular, we conclude that $L = C_{\mathrm{Sym}(\Omega)}(G)$.[3]
>
> Define a bijection $\theta : G \to G, x \mapsto x^{-1}$; of course $\theta = \theta^{-1} \in \mathrm{Sym}(\Omega)$. Then, for any $x, g \in G$,
>
> $$x^{\theta^{-1}\lambda_g\theta} = (x^{-1})^{\lambda_g\theta} = (g^{-1}x^{-1})^{\theta} = xg = x^{\rho_g},$$
>
> and we conclude that $\rho_g = \theta^{-1}\lambda_g\theta$, and hence $R = \theta^{-1}L\theta$ as required.

[3]This action is called the *right regular action of G*.
[3]This action is called the *left regular action of G*.
[3]Observe that we have proved (E3.23).