7. Minimal normal subgroups and the socle

(E7.1) Find the socle of D_{10}, A_4, S_4, $S_4 \times \mathbb{Z}$.

(E7.2) Give an example of a group that has no minimal normal subgroup.

(E7.3) If G is the direct product of a (possibly infinite) number of finite simple groups, then what is $\mathrm{soc}(G)$?

(E7.4) Give a characterization of almost simple groups in terms of their socle.

(E7.5) Suppose that G is elementary abelian. How many minimal (resp. maximal) normal subgroups does G have?

(E7.6) Let V be an elementary-abelian p-group.

(1) Let G be a group of automorphisms of V as an object from **Group**. Prove that G *acts linearly* on V, i.e. prove that G is a group of automorphisms of V as an object from $\mathbf{Vect}_{\mathbb{F}_p}$.
(2) Let G be a group of automorphisms of V as an object from $\mathbf{Vect}_{\mathbb{F}_p}$. Prove that G is a group of automorphisms of V as an object from **Group**.
(3) Conclude that $\mathrm{Aut}(V) = \mathrm{GL}(V)$, whether we consider it an object of **Group** or of $\mathbf{Vect}_{\mathbb{F}_p}$.

(E7.7) Suppose that K is finite.

- Prove that $\mathrm{soc}(\mathrm{AGL}_d(K)) \cong K^d$ and, moreover, that $\mathrm{soc}(\mathrm{AGL}_d(K))$ is a minimal normal subgroup of $\mathrm{AGL}_d(K)$.
- Suppose that $\mathrm{soc}(\mathrm{AGL}_d(K) \leq G \leq \mathrm{AGL}_d(K)$. Under what conditions is $\mathrm{soc}(\mathrm{AGL}_d(K))$ a minimal normal subgroup of G?

(E7.8) Describe the structure of $\mathrm{AGL}_1(p)$ for a prime p.

(E7.9) Suppose that G is a maximal primitive subgroup of $\mathrm{Sym}(n)$. Prove that G has a unique minimal normal subgroup (and so possibility (2) in Theorem 7.4 cannot occur).[5]

(E7.10) Suppose that K is a regular normal subgroup of G, a subgroup of $\mathrm{Sym}(n)$. Let H be the stabilizer of a point in the action on $\Omega := \{1, \dots, n\}$. Then $G = KH, K \cap H = \{1\}$ and, in particular, G splits over K, i.e. $G = K \rtimes H$.

(E7.11) Let V be a vector space. Prove that

- $N \cap \mathrm{GL}(V) = \{1\}$, where N is the set of translations of V;
- $\mathrm{Aut}(V) = N \rtimes \mathrm{GL}(V) \cong \mathrm{AGL}_d(K)$ where $d = \dim(V)$, the dimension of V as a vector space over K;
- $\mathrm{AGL}(V)$ acts faithfully and 2-transitively on V;
- The stabilizer of the zero vector is $\mathrm{GL}(V)$.

(E7.12) Let $\mathrm{AGL}_d(p) = \mathrm{Aut}(\Omega)$ for Ω an object from $\mathbf{Aff}_K$. Can you specify necessary and sufficient conditions for a subgroup $G \leq \mathrm{Aut}(\Omega)$ to act primitively on Ω.

(E7.13) Let G be a subgroup of $\mathrm{Sym}(k^\ell)$ with $k \in \{3, 4\}$, and suppose that $G \cong \mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$ in the product action. Prove that if $k \leq 5$, then G preserves an affine structure, and describe the group $\mathrm{AGL}_d(p)$ in $\mathrm{Sym}(k^\ell)$ that contains G.

(E7.14) Let k and ℓ be integers with $k \geq 5$. Show that $W := \mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$ has a unique minimal normal subgroup, and give its isomorphism type.

[5]You may find it helpful to refer to the proof of (E3.23) and (E3.24).