8. O'Nan-Scott

(E8.1) Prove that the ψ-action of T on Γ^ℓ is semiregular, and that the orbits in this action form blocks for the action of W on Γ^ℓ.

(E8.2) Suppose that G is a transitive subgroup of $\mathrm{Sym}(\Omega)$ and that $C \leq C_{\mathrm{Sym}(\Omega)}(G)$. Then the C-orbits form a set of blocks for G.

(E8.3) Prove that W acts faithfully on Ω.

(E8.4) Show that the tuple $(\tau_1, \dots, \tau_\ell)$ induces a permutation of Ω if and only if $\tau_1 = \tau_2 = \cdots = \tau_m$.

(E8.5) Prove that this action of $W \rtimes \mathrm{Aut}(T)$ on Γ^ℓ induces an action on Ω with kernel, K, isomorphic to T. Prove moreover that $G := (W \rtimes \mathrm{Aut}(T))/K$ has a normal subgroup $H \cong W$ such that $G/H \cong \mathrm{Out}(T)$.

(E8.6) Let T be a finite simple group and let

$$D := \{(t,t) \in T \times T\}.$$

Prove that D is a maximal subgroup of $T \times T$.

(E8.7) Prove this result for the case $\ell = 2$. (Recall that the action of G on the set $\{T_1, T_2\}$ is necessarily primitive in this case, so you need to prove that the group G always acts primitively.) You can do $\ell > 2$ if you want a challenge!