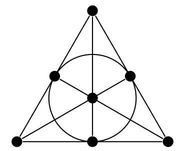

FIGURE 4. The Fano plane.

11. Projective Space

One can approach the study of projective spaces from a number of different angles. Since we will be primarily interested in projective spaces of finite dimension over finite fields, our spaces themselves are finite and so fall most naturally (it seems to me) into the sphere of combinatorics. Our approach is, therefore, combinatorial.

11.1. Incidence structures.

We define a category **IncStruct** as follows.

Objects: An object in **IncStruct** is a finite tuple $(P_1, \dots, P_\ell, I)$ where $P_1, \dots, P_\ell$ are sets, and I is a subset of $P_1 \times \cdots \times P_\ell$.

Some terminology: An object $\mathcal{I} := (P_1, \dots, P_\ell, I)$ is called an *incidence structure*. Elements from set P_i are said to *have type* i. If I contains an element $(p_1, \dots, p_\ell)$ then, for $1 \leq i, j \leq \ell$, we say that p_i *is incident with* p_j. The incidence structure is called *finite* if $P_1, \dots, P_\ell$ are all finite.

To complete the definition of this category we must, of course, define what the arrows are. We will do this shortly, but first some examples. Note that the definition of an incidence structure is extremely general; in most cases incidence structures are only studied subject to extra axioms.

Example 22. We have seen the category **SimpleGraph** earlier in lectures, and we considered the category **Graph** in exercises. Both of these categories could be seen as (full) subcategories of **IncStruct**. In particular we define the category **Graph** as follows:

Objects: An object $\mathcal{G} := (P_1, P_2, I)$ is an incidence structure with 2 types. (Elements of P_1 are what we think of as *vertices*, elements of P_2 are *edges*). We require that, for any $p_2 \in P_2$,

$$|\{p_1 \in P_1 \mid (p_1, p_2) \in I\}| \leq 2.$$

Since **Graph** is a full subcategory of **IncStruct**, the definition of arrows in **Graph** is the same as the definition in **IncStruct**, and we give this definition in a moment.

Example 23. An **abstract** *projective plane* is an incidence structure (P, L, I) such that
(1) any two elements of P are incident with a unique element of L;
(2) any two elements of L are incident with a unique element of P.
(3) there are four elements of P such that no element of L is incident with more than two of them.

We call elements of P *points* and elements of L *lines*. The third axiom listed above ('presence of a quadrangle') is there simply to eliminate some degenerate examples.

If P is finite it is easy to see that L is also finite and, in fact, that $|P| = |L|$. The smallest finite projective plane has $|P| = 7$. It is called the *Fano plane* and is represented in Figure 4; in this representation elements of P are drawn as points, and elements of L are drawn as lines (in six cases) or as a circle; each element of L is incident with three elements of P (and vice versa).

Example 24. Let V be a vector space of dimension n over a field k. We define *projective space* $\mathrm{PG}(V)$, or $\mathrm{PG}_{n-1}(k)$, to be the incidence structure $(V_1, \dots, V_{n-1}, I)$ where, for $i = 1, \dots, n-1$, V_i is the set of subspaces of V of dimension i and

$$I := \{(v_1, \dots, v_{n-1}) \in V_1 \times \cdots \times V_{n-1} \mid v_1 < v_2 < \cdots < v_{n-1}\}.$$

In other words two subspaces are incident if and only if one is contained in the other. This fact allows us to relax language a little: we say things like 'v_1 lies on v_2', or 'v_2 contains v_1' when we really mean that 'v_1 and v_2 are incident'.

A subspace $U \in V_i$ is said to have *projective dimension*, $\mathrm{pdim}(U)$, equal to $i-1$ and we call elements of V_1 'points', elements of V_2 'lines', and elements of V_{n-1} 'hyperplanes'. If k is finite of order q, then we sometimes write $\mathrm{PG}_{n-1}(q)$ for $\mathrm{PG}_{n-1}(k)$. The incidence structure $\mathrm{PG}_2(q)$ is called the *Desarguesian projective plane of order* q.

(E11.1) *Show that* $\mathrm{PG}_2(2)$ *and the Fano plane are the same incidence structure. (We would do better to write that "*$\mathrm{PG}_2(2)$ *and the Fano plane are isomorphic as incidence structures", but we have not yet defined what we mean by isomorphism.)*

(E11.2*) *Show that, for any prime power* q, $\mathrm{PG}_2(q)$ *is an abstract projective plane.*

11.2. **Some counting.** We are interested in calculating the order of $V_1, \dots, V_{n-1}$ in $\mathrm{PG}_{n-1}(q)$. To do this it is convenient to introduce *Gaussian coefficients*. Let q be a prime power, m and n positive integers. Then define

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{m-1})}{(q^m-1)(q^m-q)\cdots(q^m-q^{m-1})}. \tag{10}$$

Lemma 11.1. (1) *The number of subspaces of projective dimension* $m-1$ *in* $\mathrm{PG}_{n-1}(q)$ *is* $\begin{bmatrix} n \\ m \end{bmatrix}_q$. *In particular, the number of points in* $\mathrm{PG}_{n-1}(q)$ *is* $\begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{q^n-1}{q-1}$.

(2) *The number of subspaces of projective dimension* $m-1$ *containing a subspace of projective dimension* $l-1$ *in* $\mathrm{PG}_{n-1}(q)$ *is* $\begin{bmatrix} n-l \\ m-l \end{bmatrix}_q$.

Proof. Let V be an n-dimensional vector space over $k = \mathbb{F}_q$. To prove (a) we count the number of linearly independent m-tuples of vectors in V. The first entry in the m-tuple can be chosen to be any non-zero vector, there are q^n-1 of these; the second must lie outside the span of the first, so there are q^n-q choices for this, then q^n-q^2 for the third and so on. We conclude that the numerator of the right-hand side of (10) corresponds to the number of linearly independent m-tuples of vectors. Now the result is completed by observing (using the same reasoning) that the denominator of the right-hand side of (10) corresponds to the number of linearly independent m-tuples of vectors all lying inside any given m-dimensional subspace of V.

(E11.3) *Prove (b).*

□

Gaussian coefficients have properties resembling those of binomial coefficients, to which they tend as $q \to 1$.

(E11.4) *Prove that* $\lim_{q\to 1} \begin{bmatrix} n \\ m \end{bmatrix}_q = \binom{n}{m}$.

(E11.5) *Prove that*

$$\begin{bmatrix} n \\ m \end{bmatrix}_q + q^{n-m+1}\begin{bmatrix} n \\ m-1 \end{bmatrix}_q = \begin{bmatrix} n+1 \\ m \end{bmatrix}_q.$$

11.3. **Collineations and the fundamental theorem.** We are ready to complete the definition of the category **IncStruct**:

Arrows: Let $\mathcal{I}_1 = (P_1^1, \dots, P_\ell^1, I^1), \mathcal{I}_2 = (P_1^2, \dots, P_\ell^2, I^2)$ be incidence structures. An arrow $\mathcal{I}_1 \xrightarrow{\mathcal{F}} \mathcal{I}_2$ is a set of ℓ functions $\phi_i : P_i^1 \to P_i^2$ such that

$$(p_1, \dots, p_\ell) \in I^1 \Longrightarrow (\phi(p_1), \dots, \phi(p_\ell)) \in I^2.$$

Note that we only define arrows between incidence structures having the same number of types.

The notion of isomorphism is now clear: An *isomorphism* $\mathcal{I}_1 \xrightarrow{\mathcal{F}} \mathcal{I}_2$ is a set of ℓ bijections $\phi_i : P_i^1 \to P_i^2$ such that

$$(p_1, \dots, p_\ell) \in I^1 \Longleftrightarrow (\phi(p_1), \dots, \phi(p_\ell)) \in I^2.$$

One would expect that an isomorphism $\mathcal{I}_1 \xrightarrow{\mathcal{F}} \mathcal{I}_1$ would be known as an automorphism, however the terminology for this category is a little different: such a thing is called a *collineation*. As usual we denote the set of all collineations of an incidence structure $\mathcal{I}$ by $\mathrm{Aut}(\mathcal{I})$.

Example 25. Let V be an n-dimensional vector space over a field k. The action of $\Gamma L(V)$ on the set V extends naturally to an action on the set of subspaces of V: if $v_1, \dots, v_k \in V$ and $g \in \Gamma\mathrm{L}(V)$, then we write $U = \langle v_1, \dots, v_n \rangle$ and define

$$U^g = \langle v_1^g, \dots, v_k^g \rangle.$$

(E11.6*) *Prove that this action is well-defined, and that the action preserves the incidence relation for* $\mathrm{PG}(V)$.

This exercise implies that we have a homomorphism $\phi : \Gamma\mathrm{L}(V) \to \mathrm{Aut}(\mathrm{PG}(V))$. The next result asserts that this homomorphism is also surjective, i.e. all collineations of $\mathrm{PG}(V)$ are induced by a semilinear transformation.

Theorem 11.2. *(Fundamental theorem of projective geometry) If* $\dim V \geq 3$, *then* $\mathrm{Im}(\phi) = \mathrm{Aut}(\mathrm{PG}(V))$.

Proof. This is omitted. See [Camb, Chapter 1] or [Tay92]. □

The first isomorphism theorem of group theory implies, then, that

$$\mathrm{Aut}(\mathrm{PG}(V)) \cong \Gamma\mathrm{L}(V)/\ker(\phi).$$

(E11.7) *Prove that* $\ker(\phi) = \{\alpha I \in \mathrm{GL}(V) \mid \alpha \in k\}$.

Let us write K for $\ker(\phi)$ and observe that K is just the group of scalar transformations in $\Gamma L(V)$. Note that K is actually a subgroup of $\mathrm{GL}(V)$.

We now define three new groups in terms of this subgroup K.

(1) $\mathrm{P}\Gamma\mathrm{L}(V) := \Gamma\mathrm{L}(V)/K$;
(2) $\mathrm{PGL}(V) := \mathrm{GL}(V)/K$;
(3) $\mathrm{PSL}(V) := \mathrm{SL}(V)/(K \cap \mathrm{SL}(V))$.

For $X \in \{\Gamma, G, S\}$ we write $\mathrm{P}XL_n(k)$ as a synonym for $\mathrm{P}XL(V)$.

Observe that the Fundamental theorem of projective geometry could be expressed as follows: If $\dim V \geq 3$, then

$$\mathrm{P}\Gamma\mathrm{L}(k) = \mathrm{Aut}(\mathrm{PG}(V).$$

In particular the three groups just defined all act faithfully on $\mathrm{PG}(V)$.

(E11.8) *Prove that K is central in* $\mathrm{GL}(V)$. *Can you characterize those fields k and those vector spaces V for which K is central in* $\Gamma\mathrm{L}(V)$*?*

(E11.9*) *Prove that*

$$|\mathrm{PGL}_n(\mathbb{R}) : \mathrm{PSL}_n(\mathbb{R})| = \begin{cases} 1, & \textit{if } n \textit{ is odd;} \\ 2, & \textit{if } n \textit{ is even.} \end{cases}$$

11.4. **Duality.** Next we define a variant of the category **IncStruct** that we call **WIncStruct**. The objects are the same – incidence structures – but we allow more arrows:

Arrows: Let $\mathcal{I}_1 = (P_1^1, \dots, P_\ell^1, I^1), \mathcal{I}_2 = (P_1^2, \dots, P_\ell^2, I^2)$ be incidence structures. An arrow $\mathcal{I}_1 \xrightarrow{\mathcal{F}} \mathcal{I}_2$ in **WIncStruct** is a permutation $\pi \in S_k$, and set of ℓ functions $\phi_i : P_i^1 \to P_{i\pi}^2$ such that

$$(p_1, \dots, p_\ell) \in I^1 \Longrightarrow (\phi(p_{1\pi^{-1}}), \dots, \phi(p_{\ell\pi^{-1}})) \in I^2.$$

Note that, as in **IncStruct**, we only define arrows between incidence structures having the same number of types. More arrows are allowed here because we allow the types of objects to be jumbled.

An *isomorphism* $\mathcal{I}_1 \xrightarrow{\mathcal{F}} \mathcal{I}_2$ is a permutation $\pi \in S_k$, and set of ℓ bijections $\phi_i : P_i^1 \to P_{i\pi}^2$ such that

$$(p_1, \dots, p_\ell) \in I^1 \Longleftrightarrow (\phi(p_{1\pi^{-1}}), \dots, \phi(p_{\ell\pi^{-1}})) \in I^2.$$

Of course an isomorphism in **IncStruct** is an isomorphism in **WIncStruct**. For an incidence structure $\mathcal{I}$, we call an isomorphism $\mathcal{I} \xrightarrow{\mathcal{F}} \mathcal{I}$ in **WIncStruct** a *weak collineation*, and the set of weak collineations of $\mathcal{I}$ is denoted $\mathrm{WAut}(\mathcal{I})$.

Let V be a vector space over a field k of dimension $n \geq 3$. Then we claim that there are weak collineations of $\mathrm{PG}(V)$ that are not collineations. To see this, let $V^* = \mathrm{Hom}(V, k)$, the dual space of V.

(E11.10) *V^* is a vector space over k of dimension n.*

For $U \leq V$, define the *annihilator of* U, given by

$$U^\dagger := \{f \in V^* \mid uf = 0 \text{ for all } u \in U\}.$$

(E11.11) *$U \mapsto U^\dagger$ is a bijection between the subspaces of V and the subspaces of V^*.*

(E11.12) *$U_1 \leq U_2$ if and only if $U_1^\dagger \geq U_2^\dagger$.*

(E11.13) *If $U \leq V$, then $\dim(U^\dagger) = n - \dim(U)$ and $\mathrm{pdim}(U^\dagger) = n - 2 - \mathrm{pdim}(U)$*

Since all vector spaces over k of dimension n are mutually isomorphic, it follows that $V^* \cong V$ and so the function $V \to V^*$, $U \mapsto U^\dagger$ can be thought of as a map from V to V; indeed by (E11.12) we have a function $\mathrm{PG}(V) \to \mathrm{PG}(V)$.

(E11.14) *Prove that $U \to U^\dagger$ is a weak automorphism of $\mathrm{PG}(V)$.*

A weak collineation $\mathrm{PG}(V) \to \mathrm{PG}(V)$ with the property that a subspace of dimension d is mapped to a subspace of dimension $n - d$ is called a *duality*. Clearly $U \to U^\dagger$ is a duality. In particular, if $n \geq 3$, the function $U \to U^\dagger$ cannot be an automorphism of $\mathrm{PG}(V)$.

Proposition 11.3. *If $n \geq 3$, and Δ is a duality of $\mathrm{PG}(V)$, then $\Delta = st^{-1}$ where s is induced by a semilinear automorphism $V \to V^*$ and t is the annihilator map, $U \to U^\dagger$.*

Proof. Since Δ is a duality, the map Δt is an isomorphism $\mathrm{PG}(V) \to \mathrm{PG}(V*)$. Since $V \cong V^*$, the Fundamental Theorem of Projective Geometry implies that it is induced by a semilinear isomorphism $V \to V^*$. □

(E11.15*) *Prove that, for $n \geq 3$, $\mathrm{WAut}(\mathrm{PG}_n(q))$ contains $\mathrm{Aut}(\mathrm{PG}_n(q))$ as an index 2 subgroup. Can you say any more about the structure of $\mathrm{WAut}(\mathrm{PG}_n(q))$?*

11.5. **Abstract projective space.** It turns out that one can characterize the geometric properties of $\mathrm{PG}_{n-1}(q)$ rather straightforwardly. For $\mathcal{I} = (P_1, P_2, I)$ a finite incidence structure of points and lines we define the following three properties.

(APS1) Two points lie on a unique line.
(APS2) A line meeting two sides of a triangle, not at a vertex, meets the third also.[36]
(APS3) A line contains at least two points.
(APS4) A line contains at least three points.

An incidence structure satisfying (APS1) to (APS3) is called an *abstract projective space*. If it satisfes (APS4) then it is a *thick abstract projective space*.

(E11.16) *$\mathrm{PG}_{n-1}(q)$ is a thick abstract projective space.*

The key theorem here is the following which we will not prove.

Theorem 11.4. (Veblen-Young) *A finite thick abstract projective space $\mathcal{I} = (P_1, P_2, I)$ with $1 < |P_1|, |P_2|$ is either*

- *a projective plane, or*
- *isomorphic to $\mathrm{PG}_{n-1}(q)$ for some n and q.*

Note that the condition $1 < |P_1|, |P_2|$ is only present to eliminate some obvious and uninteresting degeneracies. The Veblen-Young theorem reduces the question of classifying the finite thick abstract projective spaces to that of classifying the finite abstract projective planes. Unfortunately the latter is a very difficult project! For instance there are many finite abstract projective planes other than $\mathrm{PG}_2(q)$.

We remark finally that the ordinary triangle is an example of an abstract projective space that is not thick. For this reason projective planes are sometimes thought of as 'generalized triangles'. This terminology will assume more significance when we come to consider polar spaces.

[36] We are using descriptive language here to save pain. It is hopefully clear what we mean by a 'triangle' and 'meeting'...