8. O'Nan-Scott

Throughout this section Ω is a finite set. There are two versions of the O'Nan-Scott theorem, the first gives the structure of the maximal subgroups of $\mathrm{Sym}(\Omega)$ and is due (unsurprisingly) to O'Nan and Scott (independently). This is the version that we will consider here.

Before we proceed, a word about the second version: this is a stronger statement outlining the structure of *all* finite primitive permutation groups – not just those that are maximal in $\mathrm{Sym}(\Omega)$. This version was stated initially by O'Nan and Scott, but it contained an error that was later corrected by Aschbacher, and hence this theorem is sometimes called the Aschbacher-O'Nan-Scott theorem.

The proof of the second, stronger statement is not much more difficult that the one we consider, save for two facts: First it needs an extra definition, that of a *twisted wreath product*, that I do not want to discuss here. Second all known proofs are dependent on a proof of the *Schreier Conjecture*, a result that is only known to be true as a consequence of the Classification of Finite Simple Groups. [34]

Throughout this section Ω is a finite set of order n.

8.1. The statement.

Theorem 8.1. *(O'Nan-Scott theorem) Let G be a maximal subgroup of $\mathrm{Sym}(\Omega)$. One of the following holds:*

1. *G is intransitive and $G = \mathrm{Sym}(k) \times \mathrm{Sym}(n-k)$ where $1 < k < n$;*
2. *G is transitive and imprimitive and $G = \mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$ where $1 < k, \ell < n$ with $n = k\ell$;*
3. *G preserves a product structure, and $G = \mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$ where $2 < k < n$ and $1 < \ell < n$ with $n = k^\ell$;*
4. *G preserves an affine structure, and $G = \mathrm{AGL}(d, p)$ where $d \geq 1$, p is prime, and $n = p^d$;*
5. *G is of diagonal type; or*
6. *G is almost simple.*

Note that the statement does not assert that all of the listed groups are maximal, but that all maximal groups are listed.

Referring to Corollary 7.9, we see that, in order to prove the O'Nan-Scott theorem, we must prove the following assertion:

Theorem 8.2. *Suppose that G is a primitive subgroup of $\mathrm{Sym}(\Omega)$, and that G contains a unique minimal normal subgroup K. Suppose that K is non-abelian. Then G either preserves a product structure, is almost simple, or is of diagonal type.*

We will not prove this theorem here, but we make one remark. Suppose that K is regular; then (E3.23) implies that $C := C_{\mathrm{Sym}(\Omega)}(K)$ is also regular. Since K is non-abelian, C is distinct from K; indeed, since K is a direct product of some number of isomorphic non-abelian simple groups, $C \cap K = \{1\}$. But this contradicts the fact that K is the unique minimal normal subgroup K.

Thus we conclude that K is not regular. Now Theorem 8.2 follows immediately from [DM96, Theorem 4.6.A].

8.2. Groups of diagonal type.

To understand the statement of the O'Nan-Scott theorem, we need to define 'subgroups of diagonal type'.

Let T be a finite non-abelian simple group of order k. By considering the right regular action of T (see Example 14), we obtain an embedding $T \leq \mathrm{Sym}(\Gamma)$ where Γ is a finite set of order k. Let Δ be a finite set of order ℓ and consider the wreath product $W := T \wr_\Delta S_\ell$ in its product action on Γ^ℓ.

Let us fix a bijection between Γ with T, so that the two sets are identified. Then the action of W on Γ is given by right multiplication of the base group:

$$(\gamma_1, \dots, \gamma_\ell)^{(t_1, \dots, t_\ell)s} = (\gamma_1 t_1, \dots, \gamma_\ell t_\ell)^s = (\gamma_{1^{s^{-1}}} t_{1^{s^{-1}}}, \dots, \gamma_{\ell^{s^{-1}}} t_{\ell^{s^{-1}}})$$

[34] The Schreier Conjecture: If K is a finite simple group, then $\mathrm{Out}(K)$ is solvable.

Proposition 6.3 implies that, since T acts regularly on Γ, W does not act primitively on Γ^m. Let us explicitly construct a nontrivial W-congruence: Consider the action ψ of T on Γ^ℓ given by

$$(\gamma_1, \dots, \gamma_\ell)^t = (t^{-1}\gamma_1, \dots, t^{-1}\gamma_\ell).$$

(E8.1) *Prove that the ψ-action of T on Γ^ℓ is semiregular, and that the orbits in this action form blocks for the action of W on Γ^ℓ.*

In fact, Exercise (E8.1) is a specific case of the following general fact (how?).

(E8.2) *Suppose that G is a transitive subgroup of $\mathrm{Sym}(\Omega)$ and that $C \leq C_{\mathrm{Sym}(\Omega)}(G)$. Then the C-orbits form a set of blocks for G.*

Now define Ω to be the set of all T-orbits (via the action ψ) on Γ^ℓ. By (E8.1), the action of T on Γ^ℓ is semiregular, and so $|\Omega| = |T|^{\ell-1}$. In addition, (E8.1) implies that W has a well-defined action on Ω, and we call this *the diagonal action* of T^ℓ.

(E8.3) *Prove that W acts faithfully on Ω.*

We call a group $G \leq \mathrm{Sym}(\Omega)$ a group *of diagonal type* if $T^\ell \leq G \leq N_{\mathrm{Sym}(\Omega)}(T^\ell)$. To fully understand such groups, then, we should understand the structure of $N_{\mathrm{Sym}(\Omega)}(T^\ell)$.

We know already that $W \leq N := N_{\mathrm{Sym}(\Omega)}(T^\ell)$. The next lemma is [DM96, Lemma 4.5B].

Lemma 8.3. *$W \trianglelefteq N$ and $N/W \cong \mathrm{Out}(T)$.*

To see how $\mathrm{Out}(T)$ enters things, observe that $\mathrm{Aut}(T)^\ell$ acts naturally on T^ℓ via

$$(t_1, \dots, t_\ell)^{(\tau_1, \dots, \tau_\ell)} = (t_1^{\tau_1}, \dots, t_\ell^{\tau_\ell}).$$

(E8.4) *Show that the tuple $(\tau_1, \dots, \tau_\ell)$ induces a permutation of Ω if and only if $\tau_1 = \tau_2 = \cdots = \tau_m$.*

The exercise implies that the action of $\mathrm{Aut}(T)$ on Γ^ℓ defined via

$$(t_1, \dots, t_\ell)^\tau = (t_1^\tau, \dots, t_\ell^\tau).$$

induces an action on Ω. Now we define an action of $W \rtimes \mathrm{Aut}(T)$ on Γ^ℓ via

$$(\gamma_1, \dots, \gamma_\ell)^{((t_1, \dots, t_\ell)s, a)} = (\gamma_{1^{s^{-1}}} t_{1^{s^{-1}}}, \dots, \gamma_{\ell^{s^{-1}}} t_{\ell^{s^{-1}}})^a = ((\gamma_{1^{s^{-1}}} t_{1^{s^{-1}}})^a, \dots, (\gamma_{\ell^{s^{-1}}} t_{\ell^{s^{-1}}})^a).$$

(Here we write a for an element of $\mathrm{Aut}(T)$.)

(E8.5) *Prove that this action of $W \rtimes \mathrm{Aut}(T)$ on Γ^ℓ induces an action on Ω with kernel, K, isomorphic to T. Prove moreover that $G := (W \rtimes \mathrm{Aut}(T))/K$ has a normal subgroup $H \cong W$ such that $G/H \cong \mathrm{Out}(T)$.*

Example 19. The smallest n for which $\mathrm{Sym}(n)$ has a subgroup of diagonal type is $n = 60$. In this case $T \cong \mathrm{Alt}(5)$ and $\ell = 2$ and the wreath product is $W \cong \mathrm{Alt}(5) \wr \mathrm{Sym}(2)$. As usual write $W = B \rtimes H$ where $B = T \times T$ and $H \cong \mathrm{Sym}(2)$.

Consider the action of $T \times T$ on the set of blocks described above, so that we obtain an embedding of $T \times T$ in $\mathrm{Sym}(60)$; write N for $N_{\mathrm{Sym}(60)}(T^2)$. Since $\mathrm{Sym}(5)$ is a split extension of $\mathrm{Alt}(5)$ we can write N in a particularly simple form, as follows: Let $(t_1, t_2)a$ be an element of W. Now all outer automorphisms of T are induced by elements from the subgroup $J := \langle (1, 2) \rangle$. What is more the natural conjugation action of J on B given by

$$(b_1, b_2)^j = (b_1^j, b_2^j)$$

commutes with the action of H on B. Thus we can write $N \cong (T \times T) \rtimes (H \times J)$ and the action of N on Ω is given by

$$(\gamma_1, \gamma_2)^{(t_1, t_2)(1, j)} = (a^{-1}\gamma_1 t_1 a, a^{-1}\gamma_2 t_2 a) \text{ and } (\gamma_1, \gamma_2)^{(t_1, t_2)(h, j)} = (a^{-1}\gamma_2 t_2 a, a^{-1}\gamma_1 t_1 a)$$

where h is the unique non-trivial element of H. (Note that we give the action on $T \times T$ and then must quotient this by the equivalence relation given by left multiplication of T.)

Consider the element $(1, 1) \in T \times T$. Write $\mathcal{B}$ for the block containing $(1, 1)$ and observe that

$$\mathcal{B} = \{(a, a) \mid a \in T\} \in \Omega.$$

Thus, an element $(t_1, t_2)(1, j) \in N$ sends $(1, 1)$ to an element of B if and only if

$$(\gamma_1, \gamma_2)^{(t_1,t_2)(1,j)} = (a^{-1}\gamma_1 t_1 a, a^{-1}\gamma_2 t_2 a) = (\delta, \delta)$$

for some $\delta \in T$. It is easy to check that this implies that $t_1 = t_2$. Similarly an element $(t_1, t_2)(h, j) \in N$ sends $(1, 1)$ to an element of $\mathcal{B}$ if and only if

$$(\gamma_1, \gamma_2)^{(t_1,t_2)(h,j)} = (a^{-1}\gamma_2 t_2 a, a^{-1}\gamma_1 t_1 a) = (\delta, \delta)$$

for some $\delta \in T$, and once again we require that $t_1 = t_2$). Thus the stabilizer in N of the block $\mathcal{B}$ is the group

$$N_{\mathcal{B}} := \{(t, t)(h', j) \mid t \in T, h' \in H, j \in J\}.$$

This group has index 60 in N, so we see that N is transitive. Suppose that $N_{\mathcal{B}} < M \leq G$ and let $g = (t_1, t_2)(h', j) \in M \setminus N_{\mathcal{B}}$. Since $H \times J \leq N_{\mathcal{B}}$, there is an element $g' = (t, t)(h', j) \in N_{\mathcal{B}}$ and now observe that $g^{-1}g' \in (M \setminus N_{\mathcal{B}}) \cap B$. But now (E8.6) below implies that $M \geq B = T \times T$ and, since $M \geq H \times J$, $M = G$. We conclude that the action is primitive.

(E8.6) *Let T be a finite simple group and let*

$$D := \{(t, t) \in T \times T\}.$$

Prove that D is a maximal subgroup of $T \times T$.

Proposition 8.4. [DM96, Theorem 4.5.A] *If $G \leq \mathrm{Sym}(\Omega)$ is a group of diagonal type, then G is primitive if and only if the action of G by conjugation on the set $\{T_1, \dots, T_\ell\}$ of minimal normal subgroups of T^ℓ is primitive.*

In particular $N_{\mathrm{Sym}(\Omega)}(T^\ell)$ is primitive for all $\ell \geq 2$.

(E8.7) *Prove this result for the case $\ell = 2$. (Recall that the action of G on the set $\{T_1, T_2\}$ is necessarily primitive in this case, so you need to prove that the group G always acts primitively.) You can do $\ell > 2$ if you want a challenge!*

8.3. **A remark on $F^*(G)$.** The O'Nan-Scott theorem is a spectacular example of the efficacy of studying the socle of a group G when one wants to understand the behaviour of G.

There is a somewhat similar object to the socle that is also worth mentioning, that of the *generalized Fitting group*, $F^*(G)$. This object was first introduced by Bender and proved to be of central importance in the proof of CFSG. Its definition resembles that of the socle, with some extra complications that yield a rather extraordinary pay-off: it turns out that $F^*(G)$ controls the structure of the whole group G in a way that the socle cannot do in general.

We won't discuss $F^*(G)$ in this course, but the keen student may like to look it up. See, for instance, [Asc00].