12. Linear groups acting on projective space

(E12.1*) A tuple of $n+1$ points in $\mathrm{PG}_{n-1}(k)$ (i.e. a tuple of $n+1$ lines in V) is said to be *special* if no n of its entries lie in a hyperplane. Write Σ_V for the set of special tuples. Prove that the action of $\mathrm{PGL}(V)$ on Σ_V is regular.

(E12.2*) Prove that $\mathrm{PSL}_n(k)$ is 2-transitive on the points of $\mathrm{PG}_{n-1}(k)$. Prove, furthermore, that $\mathrm{PSL}_n(k)$ is 3-transitive if and only if $n=2$ and every element of k is a square.

(E12.3*)Prove that this extension is split.

(E12.4*) Prove that if $n \geq 3$, then $\mathrm{SL}_n(k)$ contains a unique conjugacy class of transvections. Prove that if $n=2$, then $\mathrm{SL}_n(k)$ contains one or two conjugacy classes of transvections. Can you characterise when $\mathrm{SL}_n(k)$ contains two conjugacy classes, and describe how the subgroup Q intersects each class? (In particular you should show that each class has non-empty intersection with Q.)

(E12.5*)Prove the remaining case.

(E12.6*)Show that the set of upper-triangular matrices with 1's on the diagonal is a Sylow p-subgroup of $\mathrm{GL}_n(q)$.

(E12.7)

(1) Write down elements of order 3, 4 and 5 in the group $\mathrm{SL}_2(5)$.
(2) Write down elements of order 6, 7 and 8 in the group $\mathrm{SL}_2(7)$.
(3) (Harder). Can you write down elements of order $q-1$, p and $q+1$ in the group $\mathrm{SL}_2(q)$? Can you describe the structure of a Sylow t-subgroup of $\mathrm{SL}_2(q)$ for different t?

(E12.8) What are the orders of elements in $\mathrm{SL}_3(q)$?

(E12.9) Describe the conjugacy classes of $\mathrm{PGL}_2(q)$. Ascertain which of these classes lies in $\mathrm{PSL}_2(q)$ and list those that 'split' into more than one $\mathrm{PSL}_2(q)$-conjugacy class. Do similarly for $\mathrm{PGL}_3(q)$.

(E12.10*)Check the details of the last paragraph.

(E12.11*)Prove that $\mathrm{PSL}_3(4) \not\cong \mathrm{SL}_4(2) \cong A_8$, despite the fact that these groups have the same orders.

(E12.12) Check that this is a well-defined automorphism of $\mathrm{PSL}_n(q)$.

(E12.13*)Prove that

$$\mathrm{Aut}(\mathrm{PSL}_n(q)) \geq \begin{cases} \mathrm{P\Gamma L}_2(q), & \text{if } n=2; \\ \mathrm{P\Gamma L}_2(q) \rtimes \langle \iota \rangle, & \text{if } n \neq 3. \end{cases}$$

Hint: you need to study the natural action of, say, $\mathrm{P\Gamma L}_n(q)$ on its normal subgroup $\mathrm{PSL}_n(q)$.