13. Forms and polar spaces

(E13.1*) Prove that the left and right radicals are subspaces.

(E13.2*) Prove that if $\dim V < \infty$, then the left and right radicals have the same dimension. Give a counter-example to this assertion when $\dim V = \infty$.

(E13.3*) Check that, if β is a non-degenerate sesquilinear form, then

$$U \to U\dagger := \{x \in V \mid \beta(x,y) = 0 \text{ for all } y \in U\}$$

is a duality

(E13.4) Prove Proposition 13.4.

(E13.5*) Prove that, if k is a field, then

$$\{\lambda \in k \mid \lambda\lambda^\sigma = 1\} = \{\epsilon/\epsilon^\sigma \mid \epsilon \in k\}.$$

(E13.6*) For $x_1, \dots, x_l \in V$, define

$$[x_1, \dots, x_l] := \{y \in V \mid y^t x_1 = y^T x_2 = \dots = y^T x_l = 0\}$$

Now define Δ_0 to be the polarity of $\mathrm{PG}(V)$ that, for $x_1, \dots, x_n \in V$, does

$$\langle x_1, \dots, x_n \rangle \longleftrightarrow [x_1, \dots, x_n].$$

Prove that this is a polarity.

(E13.7) $\mathrm{Fix}(\sigma)$ and $\mathrm{Trace}(\sigma)$ are both subfields of k.

(E13.8) $\mathrm{Fix}(\sigma) = \mathrm{Trace}(\sigma)$ unless $\mathrm{char}(k) = 2$ and $\sigma = 1$, in which case $\mathrm{Trace}(\sigma) = \{0\}$.

(E13.9) Let $\mathrm{char}(k) = 2$ and suppose that k is perfect. Let β be symmetric and define

$$U := \{x \in V \mid \beta(x,x) = 0\}.$$

Then U is a subspace of dimension at least $n-1$.

(E13.10*) Fix a basis $\mathcal{B} = \{x_1, \dots, x_n\}$ for V and let $Q : V \to k$ be a quadratic form. There is a matrix A such that $Q(x) = x^T A x$. Moreover

$$A_{ij} = \begin{cases} \beta_Q(x_i, x_j), & \text{if } i < j, \\ Q(x_i), & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

(E13.11) If $\mathrm{char}(k) \neq 2$, then $Q(x) = \frac{1}{2}\beta_Q(x,x)$.

(E13.12*) Suppose that β is a symmetric, alternating bilinear form with associated matrix B with respect to some basis β. Now define the matrix A via

$$A_{ij} = \begin{cases} B_{ij}, & \text{if } i < j, \\ 0, & \text{if } i > j. \end{cases}$$

We have not defined the diagonal on the matrix A - we can set it to be anything that we choose. Now define $Q(x) = x^T A x$. Check that Q polarizes to β.

(E13.13) If $\mathrm{char}(k) \neq 2$, then β_Q is non-degenerate if and only if Q is non-degenerate.

(E13.14*) If $\mathrm{char}(k) = 2$, k is perfect, and $Q : V \to k$ is non-degenerate, then $\dim(\mathrm{Rad}(\beta_Q)) \leq 1$.

(E13.15) Any two hyperbolic lines of the same type are isomorphic (as formed spaces).

(E13.16) Suppose that U, U' (resp. W, W') are isomorphic formed spaces of the same type. Then $U \perp W$ and $U' \perp W'$ are isomorphic formed spaces.

(E13.17*) $V = V_1 \oplus W_1$ and the restriction of the form to V_1 is non-degenerate (resp. non-singular).