16. Symplectic groups

(E16.1*)Let β_1 and β_2 be non-degenerate alternating bilinear forms defined on a $2r$-dimensional vector space V over a field k. Then $\mathrm{Isom}(\beta_1)$ and $\mathrm{Isom}(\beta_2)$ (resp. $\mathrm{Sim}(\beta_1)$ and $\mathrm{Sim}(\beta_2)$) are conjugate subgroups of $\mathrm{GL}_{2r}(k)$. Furthermore $\mathrm{SemiSim}(\beta_1)$ and $\mathrm{SemiSim}(\beta_2)$ are conjugate subgroups of $\Gamma\mathrm{L}_{2r}(k)$.

(E16.2)$|\mathrm{Sp}_{2r}(k) \cap K| = \begin{cases} 2, & \text{if } \mathrm{char}(k) \neq 2; \\ 1, & \textit{otherwise}. \end{cases}$

(E16.3) Give an alternative proof of Lemma **??** by showing that

$$X^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Longleftrightarrow \det(X) = 1.$$

(E16.4*)Prove that the permutation rank is 2 if and only if G acts 2-transitively on Ω.

(E16.5) Prove that the permutation rank of G is equal to the number of orbits of G in the induced action on Ω^2.

(E16.6*) Prove that if $\beta(x, y) = 0$, then there exists z with $\beta(x, z), \beta(y, z) \neq 0$.

(E16.7*)Prove that if $\beta(x, y) \neq 0$, then there exists z with $\beta(x, z) = \beta(y, z) = 0$.

(E16.8) Prove that this extension is split.

(E16.9*)Given a transvection t, there exists $f \in V^*$ and $a \in \ker(f)$ such that

$$vT = v + (vf)a \text{ for all } v \in V.$$

(E16.10*)Prove that symplectic transvections in $\mathrm{Sp}_6(2)$ and $\mathrm{Sp}(4, 3)$ are commutators.