16. Symplectic groups

Throughout this section β is a non-degenerate alternating bilinear form on a $2r$-dimensional vector space V over a field k.

In §14.2.1 we saw that any such pair (V,β) admits a basis $B=\{e_1,f_1,\dots,e_r,f_r\}$ such that, for all $i,j=1,\dots,r$,

$$\begin{aligned}\beta(v_i,w_j)=&\delta_{ij}=-\beta(w_j,v_i),\\ \beta(v_i,v_j)=&0=\beta(w,w_j).\end{aligned}$$

This implies the following facts, which we leave as an exercise.

> **(E16.1*)** *Let β_1 and β_2 be non-degenerate alternating bilinear forms defined on a $2r$-dimensional vector space V over a field k. Then $\mathrm{Isom}(\beta_1)$ and $\mathrm{Isom}(\beta_2)$ (resp. $\mathrm{Sim}(\beta_1)$ and $\mathrm{Sim}(\beta_2)$) are conjugate subgroups of $\mathrm{GL}_{2r}(k)$. Furthermore $\mathrm{SemiSim}(\beta_1)$ and $\mathrm{SemiSim}(\beta_2)$ are conjugate subgroups of $\Gamma\mathrm{L}_{2r}(k)$.*

These facts allow us to make the following definitions. We write K for the set of invertible scalar matrices over k.

- $\mathrm{Sp}_{2r}(k)$ is the isometry group of β;
- $\mathrm{GSp}_{2r}(k)$ is the similarity group of β;
- $\Gamma\mathrm{Sp}_{2r}(k)$ is the semi-similarity group of β;
- $\mathrm{PSp}_{2r}(k)=\mathrm{Sp}_{2r}(k)/(K\cap\mathrm{Sp}_{2r}(k))$;
- $\mathrm{PGSp}_{2r}(k)=\mathrm{GSp}_{2r}(k)/K$;
- $\mathrm{P}\Gamma\mathrm{Sp}_{2r}(k)=\Gamma\mathrm{Sp}_{2r}(k)/K$

If $k=\mathbb{F}_q$ we may write $\mathrm{Sp}_{2r}(q)$ for $\mathrm{Sp}_{2r}(k)$ and likewise for the other groups.

> **(E16.2)** $|\mathrm{Sp}_{2r}(k)\cap K|=\begin{cases}2, & \textit{if } \mathrm{char}(k)\neq 2;\\ 1, & \textit{otherwise.}\end{cases}$

We can write $\mathrm{Sp}_{2r}(k)$ in terms of matrices:

(22)
$$\mathrm{Sp}_{2r}(k)=\{X\in\mathrm{GL}_{2r}(k)\mid XAX^T=X\}$$

where A can be written in one of the following ways (each is obtained from the others by permutating the basis appropriately):

$$\begin{pmatrix}0&1&&&\\-1&0&&&\\&&0&1&\\&&-1&0&\\&&&&\ddots\end{pmatrix},\quad\begin{pmatrix}0_r&I_r\\-I_r&0_r\end{pmatrix},\quad\begin{pmatrix}&&&&&1\\&&&&\iddots&\\&&&1&&\\&&-1&&&\\&\iddots&&&&\\-1&&&&&\end{pmatrix}.$$

In what follows we will study the action of $\mathrm{Sp}_{2r}(k)$ on the points of its associated polar space, with a view to applying Iwasawa's criterion to this action. Note that, since β is alternating, all 1-dimensional subspaces of V are totally isotropic, and thus they all arise as points of the associated polar space.

Lemma 16.1. $\mathrm{Sp}_2(k)\cong\mathrm{SL}_2(k)$.

Proof. Write elements of $V=k^2$ as row vectors and define

$$\beta: V\times V\to k, (x,y)\mapsto\det\begin{pmatrix}x\\y\end{pmatrix}.$$

It is clear that β is a non-degenerate alternating form. Now, if $X\in GL_2(k)$, then

$$\beta(xX,yX)=\det\begin{pmatrix}xX\\yX\end{pmatrix}=\det\left(\begin{pmatrix}x\\y\end{pmatrix}X\right)=\det\begin{pmatrix}x\\y\end{pmatrix}\det(X).$$

Thus $\beta(xX,yX)=\beta(x,y)$ if and only if $\det(X)=1$. ∎

(E16.3) *Give an alternative proof of Lemma 16.1 by showing that*

$$X^T\begin{pmatrix}0&1\\-1&0\end{pmatrix}X=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\Longleftrightarrow \det(X)=1.$$

Lemma 16.2. $|\mathrm{Sp}_{2r}(q)|=q^{r^2}\prod\limits_{i=1}^{r}(q^{2i}-1)$.

Proof. Clearly $G=\mathrm{Sp}_{2r}(q)$ acts transitively on the set of r-tuples of hyperbolic pairs that span the space. On the other hand if $g\in G$ fixes such an r-tuple, then $g=1$. Thus the action is regular and $|G|$ is equal to the number of r-tuples of hyperbolic pairs. Let us count these r-tuples.

If (v,w) is a hyperbolic pair, then the number of choices for v is $q^{2r}-1$; the number of vectors w in $V\backslash\langle v\rangle^\perp$ is $q^{2r}-q^{2r-1}$ and, of these $\frac{1}{q-1}(q^{2r}-q^{2r-1})$ satisfy $\beta(v,w)=1$.

If we fix (v,w) and continue in $\langle v,w\rangle^\perp$, which is a symplectic non-degenerate space of dimension $2r-2$ over k, then the order formula follows by induction. □

Suppose that a group G acts transitively on a set Ω and let $\omega\in\Omega$. The *permutation rank* of G is the number of orbits of G_ω on Ω.

(E16.4*) *Prove that the permutation rank is 2 if and only if G acts 2-transitively on Ω.*

(E16.5) *Prove that the permutation rank of G is equal to the number of orbits of G in the induced action on Ω^2.*

Lemma 16.3. $\mathrm{Sp}_{2r}(k)$ *acts primitively on the set of points of its polar space. If $r\geq 2$, then the permutation rank is* 3.

Proof. Witt's Lemma implies that $G=\mathrm{Sp}_{2r}(k)$ acts transitively on points. Any pair of distinct points either spans a totally isotropic 2-space, or a hyperbolic plane. By Witt's lemma, $\mathrm{Sp}_{2r}(k)$ is transitive on the pairs of each type. Thus G has three orbits in the induced action on Ω^2 (the other being on the diagonal $\{\omega,\omega)\mid\omega\in\Omega\}$), i.e. the permutation rank is 3.

We can think of a non-trivial G-congruence on Ω as a subset of Ω^2, in which case such a congruence must be a union of the diagonal and one of the other two orbits on Ω^2. We must prove that neither of these two possibilities yields an equivalence relation. The following exercises do this by showing that, whichever union we consider, the consequent relation is not transitive.

(E16.6*) *Prove that if $\beta(x,y)=0$, then there exists z with $\beta(x,z),\beta(y,z)\neq 0$.*

(E16.7*) *Prove that if $\beta(x,y)\neq 0$, then there exists z with $\beta(x,z)=\beta(y,z)=0$.*

□

Lemma 16.4. *Let $G=\mathrm{Sp}_{2r}(k)$ and $\omega\in\Omega$, the set of points of its polar space. Then*

$$G_\omega\cong Q\rtimes(\mathrm{Sp}_{2r-2}(k)\times GL_1(k))$$

where Q is an abelian group isomorphic to the additive group $(k^{2r-1},+)$.

Proof. We assume $\mathrm{Sp}_{2r}(k)$ is defined via (22) and

$$A=\begin{pmatrix} &&&&&1\\ &&&&\iddots&\\ &&&1&&\\ &&-1&&&\\ &\iddots&&&&\\ -1&&&&& \end{pmatrix}. \tag{23}$$

Thus the associated basis is $\{v_1,\dots,v_r,w_1,\dots,w_1\}$ and, since $\mathrm{Sp}_{2r}(k)$ acts transitively on the set of points of its polar space, we can take $\omega=\langle w_1\rangle$.

Now it is easy to see that $G_{\langle w_1\rangle}$ contains the following two subgroups:

$$(24)\qquad \begin{aligned} H &:= \left\{ g := \left(\begin{array}{c|ccc|c} a & 0 & \cdots & 0 & 0 \\ \hline 0 & & & & 0 \\ \vdots & & A & & \vdots \\ 0 & & & & 0 \\ \hline 0 & 0 & \cdots & 0 & a^{-1} \end{array}\right) \;\middle|\; a \in \mathbb{F}_q^*, A \in \mathrm{Sp}_{2r-2}(k) \right\}; \\ Q &:= \left\{ g := \left(\begin{array}{c|ccc|c} 1 & a_1 & \cdots & a_{2r-2} & a_{2r-1} \\ \hline 0 & & & & b_{2r-2} \\ \vdots & & I & & \vdots \\ 0 & & & & b_1 \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right) \;\middle|\; \begin{array}{l} a_1, \ldots, a_{2r-1}, b_1, \ldots, b_{2r-2} \in k, \\ b_i = \begin{cases} -a_i, & \text{if } i \leq r-1; \\ a_i, & \text{otherwise}; \end{cases} \end{array} \right\}. \end{aligned}$$

The following facts are easy to check:

(1) $Q \cap H = \{1\}$;
(2) $|G_{\langle w_1\rangle}| = |Q| \cdot |H|$;
(3) Q is isomorphic to the additive group $(k^{2r-1}, +)$;
(4) The map $H \to Sp_{2r-2}(k) \times \mathrm{GL}_1(q), g \mapsto (A, a)$ is an isomorphism.

The first two items imply that $G_{\langle w_1\rangle} = Q \cdot H$. One can easily check that H normalizes Q, and thus Q is normal in $G_{\langle w_1\rangle}$ and we conclude that $G_{\langle w_1\rangle} = Q \rtimes H$. Now the last two items complete the proof. $\square$

16.1. **Symplectic transvections.** Recall that a *transvection* on V is an element $t \in \mathrm{GL}(V)$ such that

- $\mathrm{r}k(t - I) = 1$;
- $(t - I)^2 = 0$.

.

(E16.8*) *Given a transvection t, there exists $f \in V^*$ and $a \in \ker(f)$ such that*

$$vT = v + (vf)a \text{ for all } v \in V.$$

Let β be an alternating bilinear non-degenerate form on V. A *symplectic transvection for β* is a transvection t that lies in $\mathrm{Isom}(\beta) \cong \mathrm{Sp}_{2r}(q)$. If a and f are as in the previous exercise, we have

$$\begin{aligned} \beta(vt, wt) &= \beta(v + (vf)a, w + (wf)a) \\ &= \beta(v, w) + (wf)\beta(v, a) + (vf)\beta(a, w). \end{aligned}$$

Thus t is symplectic if and only if $(wf)\beta(v, a) = (vf)\beta(w, a)$ for all $v, w \in V$. Take w such that $\beta(w, a) = 1$ and let $\lambda = wf$, then we require that $vf = \lambda\beta(v, a)$ and so a symplectic transvection for β is given by

$$v \mapsto v + \lambda\beta(v, a)a.$$

Conversely, one can check that a transvection of this form does indeed lie in $\mathrm{Sp}_{2r}(k)$.

Lemma 16.5. *The symplectic transvections generate the symplectic group $G = \mathrm{Sp}_{2r}(k)$.*

Proof. Lemma 16.1 implies that the result is true for $n = 2$. Now we induct on n. Define

$$D := \langle t \mid t \text{ is a transvection in } G\rangle.$$

Claim: D is transitive on $V\backslash\{0\}$.
Proof of claim: Let $u, v \in V\backslash\{0\}$. If $\beta(u, v) \neq 0$, then the symplectic transvection

$$(25)\qquad x \mapsto x + \frac{\beta(x, v - u)}{\beta(u, v)}(v - u)$$

carries u to v. If $\beta(u, v) = 0$, then (E16.6) implies that we can choose w such that $\beta(u, w), \beta(v, w) \neq 0$, and so we can map u to w to v.

Claim: D is transitive on the set of hyperbolic pairs in V.

Proof of claim: By the previous claim it is sufficient to prove that if (v, w_1) and (v, w_2) are hyperbolic pairs, then there exists a transvection $t \in G_v$ such that $w_1^g = w_2$. If $\beta(w_1, w_2) \neq 0$, then the following will do:

$$x \mapsto x + \frac{\beta(x, w_1 - w_2)}{\beta(w_1, w_2)}(w_1 - w_2).$$

If $\beta(w_1, w_2) = 0$, then go via $v + w_1$ as before.

Now it is sufficient to prove that any symplectic transformation g fixing a hyperbolic pair (u, v) is a product of symplectic transvections. It is easy to see that the stabilizer of (u, v) is the isometry group of $\langle u, v\rangle^\perp$ (cf. (22)), a symplectic polar space of dimension $2r - 2$. Induction now allows us to assume that g is a product of transvections in $\mathrm{Sp}_{2r-2}(q)$ and hence in $\mathrm{Sp}(q)$. $\square$

Corollary 16.6. $\mathrm{Sp}_{2r}(k) \leq SL_{2r}(k)$.

In the next lemma we will use the fact, found in the proof of Lemma 16.5, that $\mathrm{Sp}_{2r}(k)$ is transitive on hyperbolic pairs.

Lemma 16.7. *Every symplectic transvection is contained in a conjugate of the group Q defined in Lemma 16.4.*

Proof. We can use the definition for Q given by (**??**) provided we are careful to define $\mathrm{Sp}_{2r}(k)$ with respect to the matrix (23).

Now let t be a symplectic transvection and write

$$t : V \to V, v \mapsto v + \lambda\beta(v, a)a$$

where $\lambda \in k^*$ and $a \in V$. Let $w \in V$ be such that (w, a) is a hyperbolic pair. Now extending this to a symplectic basis (with w as the first element of the basis and a the last which, in particular, is consistent with (23)) and invoking Witt's lemma, we know that we can conjugate by an element g of $\mathrm{Sp}_{2r}(k)$ so that

$$t^g = \begin{pmatrix} 1 & 0 & \cdots & 0 & \lambda\beta(v,a) \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Now $t \in Q^{g^{-1}}$ as required. $\square$

Corollary 16.8. *Let Q be the group defined in Lemma 16.4. Then*

$$\mathrm{Sp}_{2r}(q) = \langle t \mid t \textit{ is a transvection}\rangle = \langle Q^g \mid g \in \mathrm{Sp}_{2r}(q)\rangle.$$

Lemma 16.9. *Symplectic transvections in $\mathrm{Sp}_{2r}(k)$ are commutators in $\mathrm{Sp}_{2r}(k)$ except if*

$$(2r, |k|) \in \{(2,2), (2,3), (4,2)\}.$$

Proof. Let t be a transvection, and let $v \in V$ such that vt is linearly independent of v. It is easy to see that $U = \langle v, vt\rangle$ is a hyperbolic plane, and that t acts trivially on $U^\perp$.

Then t acts as a transvection on U and, by Lemma 12.5, t is a commutator in $SL(U)$ provided $|k| \geq 3$. Now Lemma 16.1 implies that t is a commutator in $\mathrm{Sp}(U)$ and hence in G, as required.

To complete the proof we must deal with $|k| \leq 3$. The following exercise does that.

> **(E16.9*)** *Prove that symplectic transvections in $\mathrm{Sp}_6(2)$ and $\mathrm{Sp}(4,3)$ are commutators.*

$\square$

Corollary 16.10. $\mathrm{Sp}_{2r}(k)$ *is perfect.*

Iwasawa's criterion and the preceding results allow us to conclude our big result.

Theorem 16.11. $\mathrm{PSp}_{2r}(q)$ *is simple unless*

$$(2r, q) \in \{(2,2), (2,3), (4,2)\}.$$

We mentioned earlier that coincidences in order between simple groups, and isomorphisms between 'different' simple groups are important. The following theorem deals with all such coincidences, and isomorphisms, between $\mathrm{PSp}_{2r}(q)$ and the other simple groups that we have encountered thus far. In light of Lemma 16.1 we restrict to $r > 1$.

Proposition 16.12. *Let K and L be simple with $K = \mathrm{PSp}_{2r}(q)$ (with $r > 1$) and $L \cong \mathrm{PSL}_m(q')$ or A_m. Then $K \not\cong L$.*

It is worth dealing with the three cases listed in Theorem 16.11 for which $\mathrm{Sp}_{2r}(q)$ is not simple. Lemma 16.1 and our results for $\mathrm{SL}_2(q)$ immediately imply that

$$\mathrm{Sp}_2(2) \cong SL_2(2) \cong S_3;$$
$$\mathrm{Sp}_2(3) \cong \mathrm{SL}_2(3) \cong A_4.$$

Our final lemma deals with the remaining case.

Lemma 16.13. $\mathrm{Sp}_4(2) \cong S_6$.

Proof. Let V be a 6-dimensional vector space over $\mathbb{F}_2$. After fixing a basis for V, observe that S_6 acts on V by permuting coordinates.

Define the form $\beta(x, y) = \sum_{i=1}^{6} x_i y_i$ and write $j = (1, 1, 1, 1, 1, 1)$. Then $\langle j\rangle^\perp$ is of dimension 5 and contains j; we define $W := \langle j\rangle^\perp/\langle j\rangle$, a vector space of dimension 4.

Observe that the action of S_6 on V induces, by restriction, a faithful action on W. Furthermore the form β induces a form β_W on W, since $\beta(x, j) = 0$ for $x \in \langle j\rangle^\perp$. Since $\beta(x, x) = 0$ for $x \in \langle j\rangle^\perp$, the form β_W is alternating and one can check that it is non-degenerate.

Since S_6 preserves β_W we obtain an embedding $S_6 \leq \mathrm{Sp}_4(2)$. Since the two groups have the same order, the result follows. $\square$