Representation and Character Theory of the Small Mathieu Groups

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This is to certify that, except where specific reference is made, the work described in this project is the result of the investigation carried out by the student, and that neither this project nor any part of it has been presented, or is currently being submitted in candidature for any award other than in part for the MMath degree of the University of South Wales.

Signed .... (student)

Date ....

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Abstract

The central object of group representation theory over the complex field is the character table of a group. In this project, we attempt to calculate the character table of a group that acts sharply 5-transitively on 12 points. We obtain this result indirectly by using the fact that the Mathieu group M_{12} is the only sharply 5-transitive group, then calculating the character table of M_{12}. We also give detailed calculations of the character tables of M_9, M_{10} and M_{11}.

In order to calculate the character tables, we first construct the small Mathieu groups using the theory of transitive extensions of groups. Next, we calculate the conjugacy classes of the small Mathieu groups, a result which has not been proved in the literature. Finally, we use various techniques from group character theory to construct the character tables. To demonstrate the use of the tables, we prove that M_{11} and M_{12} are simple. All calculations have been done without the aid of a computer.

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Chapter 1

Introduction

1.1 History and Context

Representation theory is generally considered to be the study of representing algebraic objects as sets of linear transformations of vector spaces. Concretely, this amounts to writing down the elements of algebraic structures as matrices. In recent years this has seen many generalisations, notably through the use of functors from category theory. The motivation for representation theory comes from the idea of moving a complex problem to a domain that is easier to work with.

Historically, the representation theory of finite groups was developed with the same motivation: Turning problems in group theory into problems in linear algebra. The study began with Cauchy and Dedekind's work in the early 1800s on what later became character theory [Cau41, Ded85, Tau33]. However, Frobenius was the first mathematician to study the problem for non-abelian groups; he developed most of the theory in the 1890s [Fro10, Fro96a, Fro96b]. The theory was later developed by Burnside and Schur throughout the early 1900s. An in-depth discussion of the history can be found in the book 'Pioneers of Representation Theory' by Curtis [Cur99] or in an article 'The Origins of the Theory of Group Characters' by Hawkins [Haw71].

At this point, it should be noted that group representation theory naturally splits into two domains of inquiry. The first is the study of modular representation and Brauer's theory of blocks. This can be considered to be the representation theory of finite groups over a field of positive characteristic. The second field is the complex representation theory of finite groups. Here the central object is the character table of a group. This table characterises every complex representation of a group and encodes many interesting properties about the group. For example, the character table shows the existence of any normal subgroups.

Both domains are still active areas of mathematics with many unsettled conjectures, such as Brauer's height zero conjecture [Bra56], Dade's conjecture [Dad92] and the McKay conjecture [McK72]. Specifically related to the complex side, for larger groups, calculating the character table comes with great difficulty.

Throughout this project, we will develop the complex representation theory of finite groups. Our goal will be to calculate the character tables of the small Mathieu groups, two of which are part of the collection of the now famous sporadic simple groups. This collection of groups contains every finite simple group that is not the member of an infinite family of finite simple groups.

The groups referred to as the small Mathieu groups will be denoted M_9, M_{10}, M_{11} and M_{12}. In 1861, Mathieu constructed the groups M_{12} and M_{24} [Mat61, Mat73]. There was some controversy regarding his construction. Many mathematicians believed that the groups he had constructed were actually certain alternating groups and not in fact new at all. In 1897 Miller incorrectly claimed that M_{24} did not exist [Mil97], he later corrected his mistake and gave a proof that the groups were simple [Mil00].

There are now many constructions of the Mathieu groups, presenting them as automorphisms of various objects [CCN$^+$85]. The earliest is due to Witt in the 1930s, who constructed them as the automorphism groups of certain Steiner systems [Wit38a, Wit38b].

The Mathieu groups M_{11}, M_{12}, M_{22}, M_{23} and M_{24} were the first of the 26 sporadic simple groups to be discovered. That is, these were the first groups without any non-trivial normal subgroups to not belong to an infinite family. The Mathieu groups have been linked to many beautiful mathematical objects, including the binary and ternary Golay codes and the Leech lattice [MS85, CCN$^+$85, Bar93, CS99].

In 1904, Frobenius calculated the character table of M_{12} and M_{24} [Fro04]. However, many of his methods are unclear, for example, it is not mentioned how he obtained the conjugacy classes of the groups, or what properties he uses from Mathieu's constructions. Therefore, it seems desirable to recreate Frobenius's research, giving a full description of how a 19th-century mathematician could perform in-depth calculations on a group with 95040 elements.

1.2 Project Structure

The aim of this project was to calculate the character tables of the small Mathieu groups. More specifically, our initial aim was to prove the following theorem: "Let G be a sharply 5-transitive group. Then the character table of G is...". Unfortunately, a direct proof was not possible given the time constraints. Instead, we need to use the fact that M_{12} is the only sharply 5-transitive group, and then use particular facts about M_{12}.

Chapters 2 to 5 give a self-contained treatment of group representation and character theory; developing the tools needed to calculate the character tables of the small Mathieu groups. It is assumed that the reader has some familiarity with linear and abstract algebra.

Chapter 6 outlines a construction of M_9, M_{10}, M_{11} and M_{12} based on the exercises in Biggs and Whites' book [BW79, p. 33].

Chapter 7 details the calculations of the conjugacy classes of the small Mathieu groups. The treatment here begins by trying to calculate the conjugacy classes for a group which acts sharply 5-transitively on 12 points before falling back on the construction given in Chapter 6. The work in this chapter is completely original.

Finally, Chapter 8 gives a full and completely original account of the calculations of the character tables of the small Mathieu groups. One key result following from these calculations is that M_{11} and M_{12} are simple.

1.3 A Remark on Methods

Often in mathematics, the process of proving or calculating a result is very different from the way that the result is presented. This is especially true when it comes to calculating character tables for a finite group, here the process is more of an art than an exact science.

There are numerous ways of calculating irreducible characters of a finite group. However, these methods often give irreducible characters that are already known or compound characters that cannot be broken down. Throughout Chapter 8, we will note in a few places where we have calculated something we already know, or cannot use. These notes are not a complete list of the attempts to calculate the characters of M_{12}. To give a flavour, we shall list several more calculations that were performed in an attempt to find the irreducible characters of M_{12}:

- Induction from a subgroup isomorphic to $Alt(6)$, to M_{12};
- Induction from M_{10} to M_{12};
- Induction from a subgroup isomorphic to $\mathrm{AGL}_2(3)$, to M_{12};
- Tensor products of known characters at every stage;
- Restriction of larger dimensional characters from $Sym(12)$.

The calculations that are presented in Chapter 8 seem (to us) to be the most efficient route to calculating the character tables of the small Mathieu groups.

Chapter 2

Categories and Conjugacy Classes

In this chapter we will develop the notion of a group action through the use of category theory. We will then use the group action to develop several concepts from group theory; specifically, conjugacy classes and group extensions. These ideas are fundamental in later chapters and underpin all of group representation theory.

2.1 A Brief Introduction to Categories

Category theory is a relatively modern development of abstract mathematics, originally developed in Eilenberg and Maclane's seminal paper [EM45]. We will only be using a few ideas from the field and the treatment will be mostly adapted from 'An Introduction to Category Theory' by H. Simmons [Sim11].

Definition 2.1. [Gil16, p. 4] *Category*
A *category* $\mathfrak{C}$ consists of a class[1] Obj of abstract elements called *objects* and a class Arw of abstract elements called *arrows*. Each arrow has an assignment of *source*: $Arw \to Obj$ and *target*: $Arw \to Obj$; these are represented in the obvious way:

$$A \xrightarrow{f} B$$

Two arrows are composable if the first arrow's target is exactly the second arrow's source, if

[1]The idea of a class here is rather vague and deliberately so; by not defining the collection of things as a set we can skirt around Russell's paradox.

this is the case we also get a composite arrow:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow^{fg} & \downarrow g \\ & & C \end{array}$$

Finally for every object A in $\mathfrak{C}$ there exists an identity mapping 1_A such that:

$$A \xrightarrow{1_A} A$$

A *Category* must satisfy two axioms:

(C1) The triple product of arrows $f(gh)$ is defined if and only if $(fg)h$ is defined and when either is defined, it must satisfy the associative law: $f(gh) = (fg)h$.

(C2) Consider an arrow f from A to B and two compatible identity arrows as follows:

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B$$

Then we require that $f1_A = f = 1_B f$ holds[2].

A category $\mathfrak{D}$ is a *subcategory* of $\mathfrak{C}$ if the class of objects of $\mathfrak{D}$ is a subclass of the class of objects of $\mathfrak{C}$.

Example 1. *Examples of Categories*

1. The following is an exercise in 'An Introduction to Category Theory' by H. Simmons [Sim11, p. 7]. Consider a category $\mathfrak{Set}$ where the objects are sets and the arrows are functions between sets. Let X and Y be sets with identity arrows 1_X and 1_Y and let f, g and h be functions. Then the two axioms are easy to verify:

   (C1) Assume the triple product $f(gh)$ is defined with X as the source of f and let $x \in X$ then

   $$(fg)h(x) = fg(h(x)) = f(g(h(x)) = f(gh(x))$$

   (C2) Now assume that f maps X to Y then

   $$(f1_X)(x) = f(1_X(x)) = f(x) \text{ and } 1_Y f(x) = 1_Y(f(x)) = f(x)$$

2. Now consider a category $\mathfrak{Grp}$ where the objects are groups and the arrows are group homomorphisms. The axioms are again easy to verify as morphisms can be composed.

[2] As a clarification, we are applying functions on the left here and will continue to do so.

Let g, h be elements of a group G. Then we have

$$\phi\psi(g \cdot h) = \phi(\psi g \cdot \psi h) = \phi\psi(g) \cdot \phi\psi(h).$$

Every group has a trivial automorphism, hence, $\phi(1_G g) = \phi(g)$ and $1_{\phi(G)}\phi(g) = \phi(g)$.

3. An important subcategory of $\mathfrak{Grp}$ is $\mathfrak{Ab}$, the category of Abelian groups.

4. This example is adapted from 'Category Theory in Context' by Emily Riehl [Rie16, p. 5]. We can now introduce the idea of a single object category, where the category has one object $\star$ and the arrows are mapping the object to itself.

   A particularly imporant class of examples is given by considering a single group G as a category $\mathfrak{C}_G$. The category has arrows which are precisely the elements of the group. Composition is defined by composition of group elements and the identity arrow is the identity of the group. In fact we can give an alternative definition of a group in the language of categories, "a group is a single object category in which all of the arrows are isomorphisms".

After the previous example it may be worth stating exactly what we mean by an *isomorphism*. If we have a pair of arrows $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ such that $gf = 1_A$ and $fg = 1_B$, that is f and g for an inverse pair, then f and g are each called an *isomorphism*. If an arrow has an equal source and target, for example $A \xrightarrow{f} A$, then it is called an *endomorphism*. If an arrow is both an *endomorphism* and an *isomorphism* then it is an *automorphism*.

Lemma 2.2. *Given an object X in a category $\mathfrak{C}$ the set of all automorphisms of X forms a group $\mathrm{Aut}_{\mathfrak{C}}(X)$ called the automorphism group of X.*

Proof. The proof simply involves checking the axioms of a group hold. As composition of two automorphisms is an automorphism, the set is closed under composition. The axioms of identity and associativity are immediate from the definition of a category. As automorphisms are isomorphisms, every automorphism has an inverse isomorphism which is also an endomorphism. Hence, $\mathrm{Aut}_{\mathfrak{C}}(X)$ is a group under composition. □

Example 2. *Automorphism Groups*
Given a finite set Δ, the automorphism group in $\mathfrak{Set}$ will be $Sym(\Delta)$. For an n-dimensional vector space V over a field $\mathbb{F}_q$, where $q = p^k$, the automorphism group in $\mathfrak{Vect}_{\mathbb{F}_q}$ (the category of vector spaces over a field $\mathbb{F}_q$) is $\mathrm{GL}_n(q)$. However, $\mathrm{Aut}_{\mathfrak{Grp}}(V) = \mathrm{GL}_{kn}(p)$.

The automorphism group in the category $\mathfrak{Graph}$ of a regular n-gon is isomorphic to D_{2n}, the dihedral group on n points. However, in the category $\mathfrak{Set}$ the automorphism group will be $Sym(n)$.

Clearly, the choice of category is important. Another example of this is the automorphism group of a group in $\mathfrak{Grp}$ is $\mathrm{Aut}(G)$ but in $\mathfrak{Set}$ it is $Sym(G)$. A specific example of this is $\mathrm{Aut}_{\mathfrak{Grp}}(C_p) = C_{p-1}$ for p a prime, whereas $\mathrm{Aut}_{\mathfrak{Set}}(C_p) = Sym(p)$.

Definition 2.3. [Gil16, p. 5] *Hom-Space*
In a category $\mathfrak{C}$ the class of morphisms from an object X to an object Y is called the *Hom-Space* and is denoted $\mathrm{Hom}_{\mathfrak{C}}[X, Y]$.

We should note that all of the categories we will be working with are termed 'locally small'. This means that between any two objects there is only a sets worth of morphisms, which in turn means we can treat a Hom-space as a set. One last idea we will use from category theory is the idea of a *Short Exact Sequence*. First, we define the *zero object* to be the trivial object in a category $\mathfrak{C}$ and we will denote this by 0, or if the category is $\mathfrak{Grp}$ then by $\{1\}$.

$$0 \xrightarrow{f_1} N \xrightarrow{f_2} A \xrightarrow{f_3} Q \xrightarrow{f_4} 0 \tag{2.1}$$

Definition 2.4. [Hat02, p. 113] *Short Exact Sequence*
The sequence of objects and arrows in (2.1) is called a *Short Exact Sequence* if $\mathrm{Ker}(f_{i+1}) = \mathrm{Im}(f_i)$. We describe the sequence as *split* if there exists an arrow $g : Q \to A$ such that $f_3 g = 1_Q$, where 1_Q is the identity arrow of Q.

Example 3. *Groups and Short Exact Sequences*
In the category $\mathfrak{Grp}$ anytime the fundamental theorem of homomorphisms can be applied to a group G we get a short exact sequence. Let G have a normal subgroup N such that $G/N \cong Q$; then the following diagram is a short exact sequence. Here τ is the inclusion map and π is the projection map.

$$\{1\} \longrightarrow N \xrightarrow{\tau} G \xrightarrow{\pi} Q \longrightarrow \{1\}$$

We shall consider some specific examples of this. First consider the dihedral group on 4 points D_8, it is easy to check that D_8 has a cyclic normal subgroup of size 4. From this we can construct the following short exact sequence.

$$\{1\} \longrightarrow C_4 \longrightarrow D_8 \longrightarrow C_2 \longrightarrow \{1\}$$

This sequence is split, however there are examples in $\mathfrak{Grp}$ that do not split. C_4 is the smallest group that does not split and Q_8 (the Quaternion group) is the smallest non abelian group that does not split. We will discuss some of the necessary conditions for a sequence to split in Section 2.3.3.

2.2 Group Actions

Parts of this section have been adapted from [Gil16], notably the definition and properties of actions and G-Sets, and Theorem 2.8.

Definition 2.5. [Gil16, p. 9] *Group Actions and G-Sets*
A *(left) action* of G on Δ is a function $\varphi : G \times \Delta \to \Delta$, where we write the image of (g, δ) as $g\delta$. We require φ to satisfy two axioms:

(A1) $\forall \delta \in \Delta$, $1_G \cdot \delta = \delta$, (where 1_G denotes the identity of G);
(A2) $\forall g, h \in G$ and $\forall \delta \in \Delta$, $(g \cdot h)\delta = g(h \cdot \delta)$.

A *G-Set* (G, Δ, φ) is a triple containing a group G, a set Δ and an action of G on Δ.

Example 4. *Group Actions*

1. Consider $Sym(n)$ acting on a set $X = \{1, \ldots, n\}$. A 'natural' action of $Sym(n)$ is to simply apply a permutation $g \in Sym(n)$ to an element $x \in X$. Formally we can define $\varphi : (g, x) \mapsto gx$. Now take $x = 1$ and $g = (317)$ then $\varphi((317), 1) = (317) \cdot 1 = 7$.

2. Now consider the dihedral group on 6 points, $D_{12} = \langle(123456), (12)(36)(45)\rangle$ acting on a hexagon with each vertex labeled 1 to 6. The permutation (123456) corresponds to a rotation of $\frac{\pi}{3}$ radians and (12)(36)(45) corresponds to a flip. Simply applying the permutation to the vertices will rotate or reflect the shape.

The following Lemma was an exercise in [Gil16, p. 6].

Lemma 2.6. *A class of objects consisting of G-Sets and a class of arrows represented by a pair (α, f), where α is a group homomorphism and f is a total function, forms a category called $\mathfrak{GSet}$.*

Proof. To begin the proof we need to show how arrow composition works. Consider the following commutative diagram:

$$\begin{array}{ccccc} G \times \Delta & \xrightarrow{(\alpha,f)} & H \times \Omega & \xrightarrow{(\eta,g)} & K \times \Lambda \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\theta} \\ \Delta & \xrightarrow[f]{} & \Omega & \xrightarrow[g]{} & \Lambda \end{array}$$

The composite arrow mapping $G \times \Omega$ to $K \times \Lambda$ is given by $(\eta, g)(\alpha, f) = (\eta\alpha, gf)$. Now we can show the associativity axiom (C1):

$$\begin{aligned}(\tau, h)((\eta, g)(\alpha, f)) &= (\tau, h)(\eta\alpha, gf) \\ &= (\tau\eta\alpha, hgf) \\ &= ((\tau\eta)\alpha, (hg)f) \\ &= (\tau\eta, hg))(\alpha, f) \\ &= ((\tau, h)(\eta, g))(\alpha, f)\end{aligned}$$

Each object (G, Δ, φ) has an identity arrow $(1_G, 1_\Delta)$. Now, the identity axiom amounts to checking the identity axiom (C2) for $\mathfrak{Grp}$ and $\mathfrak{Set}$ simultaneously. As these have been individually checked in Example 1, the result follows immediately. □

Of course, groups can act on more than just sets. If we replace the set Δ in Definition 2.5 with an object[3] X from a category $\mathfrak{C}$, the notion of a group action still holds. In fact, by carefully picking the category $\mathfrak{C}$ we will be able to find 'structure preserving actions' on some objects.

Definition 2.7. *Groups acting on objects*
Let G be a group and let X be an object in a category $\mathfrak{C}$. Then G *acts on* X *as an object in* $\mathfrak{C}$ if every group element induces an arrow in $\mathfrak{C}$ and these arrows are automorphisms of X in $\mathfrak{C}$.

The following theorem was an exercise in [Gil16].

Theorem 2.8. *Let (G, X, φ) be a G-Set with G acting on X as an object in a category $\mathfrak{C}$ by an action φ. Define the following functions:*

$$\varphi^*_g\colon X \to X \text{ by } x \mapsto g(x);$$

$$\psi^\dagger\colon G \times X \to X \text{ by } (g, x) \mapsto \psi(g)(x).$$

Then the function $\varphi^* : G \to \mathrm{Aut}_{\mathfrak{C}}(X)$ by $g \mapsto \varphi^*_g$ is a group homomorphism. Conversely, given a group homomorphism $\psi : G \to \mathrm{Aut}_{\mathfrak{C}}(X)$, then the function $\psi^\dagger : G \times X \to X$ is an action of G on X as an object in $\mathfrak{C}$. Moreover, $(\varphi^*)^\dagger = \varphi$ and $(\psi^\dagger)^* = \psi$.

[3]The only objects we will be considering are 'structured sets'; that is, sets with additional structure. The additional structure could be through operators or relations on elements of the set. Examples of these include groups, rings, vectors spaces and graphs.

Proof. Since the map φ_g^* is an arrow in $\mathfrak{C}$ for every g then it is immediate that φ_g^* and $\varphi_{g^{-1}}^*$ are inverse pairs of isomorphisms. In particular, $\varphi^* : G \to \mathrm{Aut}_{\mathfrak{C}}(X)$ is well defined. Now consider

$$\varphi^*(gh) = \varphi_{gh}^* = (x \mapsto gh \cdot x) = (x \mapsto g(h \cdot x)) = \varphi_g^* \varphi_h^* = \varphi^*(g)\varphi^*(h).$$

Thus, φ^* is a group homomorphism.

Conversely, given $\psi : G \to \mathrm{Aut}_{\mathfrak{C}}(X)$ we need to check the properties of an action for $\psi^\dagger$. We know that $\psi(1) = 1 \in \mathrm{Aut}_{\mathfrak{C}}(X)$ hence $\forall x \in X$ we have $\psi^\dagger(1, x) = 1 \cdot x = x$. Now take $g, h \in G$, then,

$$gh \cdot x = \psi^\dagger(gh, x) = \psi(gh) \cdot x = \psi(g)\psi(h) \cdot x = \psi(g)(\psi(h) \cdot x) = g(h \cdot x)$$

Thus, $\psi^\dagger$ is an action of G on X as an object in $\mathfrak{C}$. Moreover,

$$(\varphi^*)^\dagger(g, x) = \varphi^*(x \mapsto g \cdot x) = \varphi_g^* = \varphi(g, x)$$

and

$$(\psi^\dagger)^*(g) = \psi^\dagger(\varphi_g^*) = \psi(x \mapsto g \cdot x) = \psi(g).$$

□

The theorem we have just proved is fundamental to the study of group actions and has an immediate consequence, if G has an action on an object X in a category $\mathfrak{C}$ then we immediately get a homomorphism into $\mathrm{Aut}_{\mathfrak{C}}(X)$. For example, if G is acting on a finite dimensional vector space V in $\mathfrak{Vect}_{\mathbb{F}}$, we immediately get a homomorphism of G into $\mathrm{GL}(V)$.

Now we have laid the framework for group actions, we are at liberty to discuss some of the properties these actions can have. Consider a G-Set (G, Δ, φ) then:

- The set of elements from G that fix a $\delta \in \Delta$ is called the *stabiliser* of δ in G. Symbolically $G_\delta = \{g \in G \colon g \cdot \delta = \delta\}$. Note that G_δ forms a subgroup of G.

- The set of elements from G that fix every element of Δ is called the *kernel* of the action, denoted by $\mathrm{Ker}(\varphi) = G_{(\Delta)} = \{g \in G \colon g \cdot \delta = \delta \ \forall \delta \in \Delta\}$. This kernel is exactly the same as the kernel of the corresponding homomorphism given by Theorem 2.8; by the fundamental theorem of homomorphisms it is a normal subgroup of G.

- The *orbit* of δ under G is given by $\delta^G = \{g \cdot \delta \colon g \in G\}$; this is exactly the set of elements in Δ that elements of G can move δ to. Note, the set of orbits partitions Δ.

An action is described as:

- *Transitive*, if the action only has one orbit; that is $\delta^G = \Delta$. Alternatively $\forall \delta_i, \delta_j \in \Delta$ there exists $g \in G$ such that $g \cdot \delta_i = \delta_j$.

- *Faithful*, if the only element of G that fixes every element of Δ is the identity, symbolically $\forall g \neq 1_G \in G$ there exists $\delta \in \Delta$ such that $g \cdot \delta \neq \delta$.

- *Semi-regular*, if the only element of $g \in G$ that fixes $\delta \in \Delta$ for all $\delta \in \Delta$, is the identity of G. Alternatively, if $\delta \in \Delta$ and $g \cdot \delta = \delta$ then $g = 1$.

- *Regular*, if it is both *transitive* and *semi-regular*.

Theorem 2.9. Orbit Stabiliser Theorem
Let G be a group acting on a finite object X in a category $\mathfrak{C}$ and let $x \in X$, then $|G| = |x^G||G_x|$.

For a proof the reader is referred to any standard text on group theory such as Durbin's book 'Modern Algebra' [Dur09, p. 245].

Example 5. *Some actions of $Sym(4)$*
We will now consider some of the actions of $G = Sym(4)$ and some of its subgroups. Let $\Delta = \{1, 2, 3, 4\}$ and consider $C_4 = \langle(1234)\rangle \leq Sym(4)$. We shall calculate 1^{C_4}, the orbit of 1.

$$1^1 = 1 \quad (1234) \cdot 1 = 2 \quad (13)(24) \cdot 1 = 3 \quad (1432) \cdot 1 = 4$$

Clearly, the orbit of 1 is Δ, therefore C_4 acts *transitively* on Δ. Evidently the only stabiliser of any element in Δ is the identity of C_4, hence the action is *semi-regular* and, as it is also transitive, we see that it is *regular* as well.

Next, consider $C_3 = \langle(123)\rangle \leq Sym(4)$ acting on Δ. A quick check will show the action has a trivial kernel and has orbits $\{1, 2, 3\}$ and $\{4\}$. We can verify Theorem 2.9 for both orbits: $3 = 1 \times 3$ and $3 = 3 \times 1$.

Now we shall look at two actions of $Sym(4)$ on Δ The first one is the same action we have used throughout this example. As C_4 is a subgroup and the action of C_4 is regular, it follows that the action of $Sym(4)$ is transitive. However, it is not semi-regular; consider the stabiliser of the point 4, a simple calculation gives $G_4 = \{1, (12), (13), (23), (123), (132)\} \cong Sym(3)$. Again, we can verify Theorem 2.9 as $24 = 4 \times 6$.

Recall that every permutation can be expressed as a finite product of transpositions. We can then define the following homomorphism

$$Sgn : Sym(4) \to C_2 \text{ by } g \mapsto \begin{cases} 1 & \text{if } g \text{ is a product of an even number of transpositions;} \\ (12) & \text{if } g \text{ is a product of an odd number of transpositions.} \end{cases}$$

This homomorphism can be turned into an action using Theorem 2.8. Define:

$$Sgn^{\dagger}\colon G \times \Delta \to \Delta \text{ by } Sgn^{\dagger}(g, \delta) \mapsto Sgn(g)\delta$$

The orbits of this action are $\{1,2\}$, $\{3\}$ and $\{4\}$. We find that $\mathrm{Ker}(Sgn^{\dagger}) = G_{(\Delta)} = \{1, (123), (132), (124), (134), (143), (142), (234), (243), (12)(34), (13)(24), (14)(23)\} \cong Alt(4) \trianglelefteq Sym(4)$.

Definition 2.10. [Gil16, p. 24] *Multiply Transitive Group*
Let, G be a group acting on a set $\Delta = \{1, \ldots, n\}$, then G acts on Δ^k by $g(\delta_1, \ldots, \delta_k) = (g(\delta_1), \ldots, g(\delta_k))$. Define,

$$\Delta^{(k)} = \{(\delta_1, \ldots, \delta_k)\colon \delta_i \neq \delta_j \ \forall i, j = 1, \ldots, k \text{ with } i \neq j\}$$

then if G acts transitively on $\Delta^{(k)}$ we say that G is *k-transitive* and acts k-transitively on Δ. Moreover, if G acts regularly on $\Delta^{(k)}$ then we say G is *sharply* k-transitive. If $k \geq 2$ we describe G as a *multiply transitive group*.

We remark that for $k \geq 2$, a k-transitive group is $(k-1)$-transitive; this can be observed by considering the fact the tuple $\delta = (\delta_1, \ldots, \delta_k)$ must be sent to every other possible tuple in $\Delta^{(k)}$. It follows that there exist elements in G that send δ to every $(\alpha_1, \ldots, \alpha_{k-1}, \delta_k) \in \Delta^{(k)}$.

Example 6. *Multiply Transitive Groups*
$Sym(n)$ acts n-transitively on a set of size n in the category $\mathfrak{Set}$. In the same category on the same sized set, $Alt(n)$ acts $(n-2)$-transitively.

We will revisit the idea of multiply transitive groups when we construct the small Mathieu groups in Chapter 6.

2.3 Groups Acting on Groups

2.3.1 Conjugacy Classes

So far, we have only looked at examples of groups acting on sets. We know groups can act on many objects in different categories as demonstrated by Theorem 2.8. In this section we will be considering groups acting on groups, specifically groups acting on themselves. First, we consider two important actions of a group acting on itself.

Definition 2.11. [Gil16, p. 13] *(Left) Regular Action*
Let G be a group acting on itself in the category $\mathfrak{Set}$ with following action; $G \times G \to G$ by

$(g, h) \mapsto gh$. We call this action the *(left) regular action* of G and it gives an embedding of G into $Sym(G)$.

Definition 2.12. [Gil16, p. 13] *The Conjugation Action*
Let G be a group acting on itself in the category $\mathfrak{Grp}$ with the following action; $G \times G \to G$ by $(g, h) \mapsto ghg^{-1}$. We call this action *conjugation* and it gives a homomorphism from G to $Aut_{\mathfrak{Grp}}(G)$.

The orbits of the conjugation action partition G. We call these partitions the *conjugacy classes* of G. If g and h are in the same conjugacy class, then they are described as being *conjugate*. The kernel of the conjugation homorphism is the centre of G, denoted $\mathrm{Z}(G)$.

The quotient $G/\mathrm{Z}(G)$ is called the *inner automorphism group* and is denoted $\mathrm{Inn}(G)$. Any other automorphisms of G are called *outer automorphisms*. The quotient group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the *outer automorphism group* and is denoted $\mathrm{Out}(G)$.

This action can also be extended to the set of subgroups of G. We describe two subgroups H and K as *conjugate subgroups* if there exists a $g \in G$ such that $gHg^{-1} = K$. We note two important instances of this. If G acts transitively on a set Ω, then the point stabilisers of G are conjugate. Secondly, all Sylow p-subgroups of G are conjugate (for more on the Sylow theorems, see Section 2.3.2).

Theorem 2.13. [Gil16, p. 14] *Let G be a group and define $\phi : G \to \mathrm{Aut}_{\mathfrak{Grp}}(G)$ by $g \mapsto \phi_g$ where $\phi_g(h) = ghg^{-1}$ (this is the homomorphism corresponding to the conjugation action) then*

1. $\mathrm{Im}(\phi) = \mathrm{Inn}(G)$;
2. $\mathrm{Ker}(\phi) = \mathrm{Z}(G)$ *where* $\mathrm{Z}(G)$ *is the centre of the group;*
3. $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$.

A proof of this theorem is given in most standard texts on group theory, for example Rotman's 'An Introduction to the Theory of Groups' [Rot95, p. 156].

Definition 2.14. *Centraliser*
Let G be a group, then the centraliser of an element $g \in G$ is the set of elements that commute with g. Let $g \in G$, then the centraliser of g in G is given by $C_G(g) = \{h \in G \colon gh = hg\}$ and note that it forms a subgroup of G. This is exactly the stabiliser of g under the conjugacy action. We can extend this to a subset S of G. Here, $C_G(S) = \{g \in G \colon gs = sg, \ \forall s \in S\}$.

Definition 2.15. *Normaliser*
Let G be a group and H be a non empty subset of G, then the *normaliser* of H in G is $N_G(H) = \{g \in G \colon gH = Hg\}$. Note that $C_G(H) \leq N_G(H)$. This is exactly the stabiliser of the set H under the conjugacy action.

Theorem 2.16. [Rot95, p. 156] *Let G be a group with a subgroup H then the following hold:*

1. $C_G(H) \trianglelefteq N_G(H)$*;*
2. *The quotient group* $N_G(H)/C_G(H)$ *embeds into* $\mathrm{Aut}(G)$*.*

A proof of this theorem is also given in Rotman's 'An Introduction to the Theory of Groups' [Rot95, p. 156].

Example 7. *Conjugacy Classes of Sym(4)*
We shall calculate the orbits of the conjugacy action of $Sym(4)$ on itself, first we shall observe that $1^{Sym(4)} = 1$ giving us our trivial class. Now we shall calculate $(12)^{Sym(4)}$ (that is the orbit of (12)):

$$\begin{aligned}
1(12)1 &= (12)\\
(13)(12)(13) &= (23)\\
(14)(12)(14) &= (24)\\
(23)(12)(23) &= (13)\\
(243)(12)(234) &= (14)\\
(24)(13)(12)(24)(13) &= (34)
\end{aligned}$$

Any further conjugations of (12) give the same six permutations, hence the orbit of (12) is $\{(12), (13), (14), (23), (24), (34)\}$. Note that Theorem 2.9 holds as $24 = 6 \times 4$. Proceeding in the same manner, we can calculate all of the remaining conjugacy classes (Table 2.1).

Elements	Class Size
1	1
(12),(13),(14),(23),(24),(34)	6
(123),(132),(124),(142),(134),(143),(234),(243)	8
(12)(34),(13)(24),(14)(23)	3
(1234),(1432),(1423),(1324),(1243),(1342)	6
Total	24

Table 2.1: The conjugacy classes of $Sym(4)$.

Notice that each class divides the order of the group, this is a consequence of Theorem 2.9. Observe that the union of the classes containing the three cycles, the double transpositions and the trivial class is actually $Alt(4)$. As $Sym(4)$ has trivial centre we can see that $\mathrm{Inn}(Sym(4)) \cong Sym(4)$.

In fact, in $Sym(n)$ the conjugacy classes are determined by cycle type; we will write these as a *partition* of n. For example, the conjugacy classes of $Sym(4)$ are, 1^4, $1^2 2^1$, 2^2, $1^1 3^1$, 4^1. For the alternating groups we have the following theorem.

Theorem 2.17. The Splitting Criteria
Given an even conjugacy class of $Sym(n)$, then either:

1. *This class is equal to a single conjugacy class of $Alt(n)$;*
2. *This class splits into two conjugacy classes of $Alt(n)$.*

The second case occurs if the partition of the class contains neither an odd number, nor a repeated number.

We omit the proof of this theorem, however, a proof of it can be found in John Wilson's book 'The Finite Simple Groups' [Wil09].

2.3.2 The Sylow Theorems

The following set of standard results are collectively known as the Sylow Theorems. They will be stated without proof; however several proofs are available and can be found in the original article by Sylow [Syl72] or in 'The early proofs of Sylow's theorem' by Waterhouse [Wat80].

Definition 2.18. *Sylow p-Subgroups*
Let G be a group with order $|G| = p^k m$ where p is a prime and p does not divide m, then a subgroup of order p^k is termed a *Sylow p-subgroup*. The set of all Sylow p-subgroups for a given prime p is denoted by $Syl_p(G)$.

Theorem 2.19. *Let G be a group of order $|G|$ then for every prime factor p with multiplicity k of $|G|$, there exists a subgroup of G of order p^k.*

Theorem 2.20. *Let G be a group of order $|G|$ and let p be a prime that divides $|G|$ then for every $H, K \in Syl_p(G)$ there exists a $g \in G$ such that $H = gKg^{-1}$; that is any two Sylow p-subgroups are conjugate.*

Theorem 2.21. *Let G be a group of order $|G|$, let p be a prime that divides $|G|$, let n_p be the number of Sylow p-subgroups of G and let $P \in Syl_p(G)$; then the following hold:*

1. *n_p divides $|G : P|$;*
2. *$n_p \equiv 1 \pmod{p}$;*
3. *$n_p = |G : N_G(P)|$.*

2.3.3 Group Extensions

A key fact in group theory is that every group G can be constructed by joining together simple groups. The study of finding every way to combine groups K and N is called the extension problem. Without going into too much detail we will describe the two types of extension, split and non split (for further information on group extensions the interested reader is recommended Brown's 'Cohomology of Groups' [Bro82]).

Definition 2.22. [Rot95, p. 40] *Direct Product*
Let H and K be groups and define $G = H \times K$ where $H \times K$ is the Cartesian product of H and K. That is $H \times K = \{(h, k)\colon h \in H, k \in K\}$. Define a multiplication on $H \times K$ by

$$(h_1, k_1)(h_2, k_2) = (h_1h_2, k_1k_2)$$

Then G is the *direct product* of H and K.

Example 8. *Some useful direct product groups*
The Klein-4 group V_4 is isomorphic to $C_2 \times C_2 = C_2^2$. Any n-dimensional vector space V over a field $\mathbb{F}$ is isomorphic to the direct product of n copies of the additive group of $\mathbb{F}$, that is $V \cong (\mathbb{F}_+)^n$.

Definition 2.23. [Rot95, p. 167] *Semi-direct Product*
Let N and K be groups and define $G = N \times K$ where $N \times K$ is the Cartesian product of N and K. Let $\varphi\colon K \to \mathrm{Aut}(N)$ by $k \mapsto \varphi_k$ be a group homomorphism. Let φ_k denote the image of k in $\mathrm{Aut}(N)$ and define a group multiplication on $N \times K$ by

$$(n_1, k_1)(n_2, k_2) = (n_1\varphi_{k_1}(n_2), k_1k_2).$$

Then G is the *semi-direct product* of N and K with respect to φ; we denote this $N \rtimes_\varphi K$.

We shall make the following remarks without proof. Let $G = N \rtimes_\varphi K$ then $N \trianglelefteq G$, $G/N \cong K$ and $|G| = |N| \times |K|$. If φ is trivial then $G \cong N \times K$. Finally, every semi-direct product gives a short exact sequence that splits as follows.

$$\{1\} \longrightarrow N \longrightarrow G \longrightarrow K \longrightarrow \{1\}\ .$$

In fact every group that can be expressed as a short exact sequence that splits has a semi-direct product structure.

Example 9. *Some useful semi-direct product groups*
Any direct product is a semi-direct product with a trivial action. The Symmetric group on

3 points, $Sym(3) \cong C_3 \rtimes C_2$ where $C_2 \cong \mathrm{Aut}_{\mathfrak{Grp}}(C_3)$). The Dihedral group on n points, $D_{2n} \cong C_n \rtimes C_2$.

If a group cannot be expressed as a semi-direct product, we describe the group as *non-split* and denote this as $N^{\cdot}K$. We still have a short exact sequence, but it does not split.

Example 10. *The Quaternion Group*
The Quaternion group Q_8 is usually defined by the following relations:

$$\langle i, j, k \colon i^2 = j^2 = k^2 = ijk,\ (ijk)^2 = 1\rangle.$$

Essentially, Q_8 is generated by three distinct order 4 elements which all square to give the same element. However, it can also be described as the non split extension $C_4^{\cdot}C_2 = (C_2^{\cdot}C_2)^{\cdot}C_2$. We will now calculate the conjugacy classes of Q_8 (Table 2.2).

Name	Elements	Class Size
$C1$	1	1
$C2$	ijk	1
$C4_A$	i, i^3	2
$C4_B$	j, j^3	2
$C4_C$	k, k^3	2

Table 2.2: The conjugacy classes of Q_8.

2.4 Group Actions and Vector Spaces

Example 11. *Groups Acting on Vector Spaces*
Recall that the general linear group of a vector space V over a field $\mathbb{F}$ is the set of all invertible linear transforms $V \to V$. A 'natural' left action of $\mathrm{GL}(V)$ on V as an object in $\mathfrak{Vect}_{\mathbb{F}}$ is to simply apply the linear transform associated with an element g of $\mathrm{GL}(V)$ to a column vector $\mathbf{v} \in V$; that is: $g \cdot \mathbf{v}$.

Now, consider the following G-Set, (C_3, V, φ) where $V \cong \mathbb{R}^3$ and $C_3 = \{1, (123), (132)\}$. Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis for V and define a left action $\varphi : C_3 \times V \to V$ by $\varphi : (g, \mathbf{e}_i) \mapsto \mathbf{e}_{g \cdot i}$. Clearly the identity permutation sends $\mathbf{e}_i$ to $\mathbf{e}_i$, now we just need to check (A2):

$$g(h(\mathbf{e}_i)) = g(\mathbf{e}_{h \cdot i}) = \mathbf{e}_{gh \cdot i} = gh(\mathbf{e}_i).$$

Clearly, the action is linear, so we conclude that this is a well-defined action. Now, observe that this action of C_3 is cyclically permuting the basis vectors of V. In fact we can even write down matrices which correspond to this action. Define $\phi : C_3 \to \mathrm{GL}_3(\mathbb{R})$ by:

$$\phi(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \phi((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \phi((132)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What we have just calculated is a *faithful* action of C_3 on V as an object in $\mathfrak{Vect}_{\mathbb{R}}$.

Another immediate consequence of Theorem 2.8 is that when a group acts on a vector space V as an object in $\mathfrak{Vect}_{\mathbb{F}}$ we immediately get a homomorphism into $\mathrm{Aut}_{\mathfrak{Vect}_{\mathbb{F}}}(V)$. If V is finite dimensional over a field $\mathbb{F}$ then $\mathrm{Aut}_{\mathfrak{Vect}_{\mathbb{F}}}(V) \cong \mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{F})$. The study of these homomorphisms is called *group representation theory.* We will conclude this chapter with an example demonstrating how groups can be constructed from representations.

Example 12. *Affine Groups*
Let V be an object in $\mathfrak{Vect}_{\mathbb{F}}$; as V is an Abelian group, any group G which acts on a vector space as an object in $\mathfrak{Vect}_{\mathbb{F}}$ can be used to construct a semi-direct product group $V \rtimes_\eta G$. Here, we could also act as an object in $\mathfrak{Grp}$ and we would construct different groups. Acting on an object in $\mathfrak{Vect}_{\mathbb{F}}$ is a stronger property than acting on an object in $\mathfrak{Grp}$, because an action on an object in $\mathfrak{Vect}_{\mathbb{F}}$ must be $\mathbb{F}$-linear as well a group automorphism.

If $G = \mathrm{GL}(V)$ then the semi-direct product group we obtain is the affine general linear group $\mathrm{AGL}(V)$. If V is of dimension n over a field $\mathbb{F}$ then $\mathrm{AGL}(V)$ is denoted $\mathrm{AGL}_n(\mathbb{F})$.We will now consider a specific example of this construction and investigate some groups which will be useful later on.

Let $V = \mathbb{F}_3^2$ so $\mathrm{Aut}_{\mathfrak{Vect}_{\mathbb{F}_3}}(V) = \mathrm{GL}_2(3)$. Our semi direct product group is $\mathrm{AGL}_2(3)$. We shall examine the subgroups of $\mathrm{GL}_2(3)$, then restrict the action on V as an object in $\mathfrak{Vect}_{\mathbb{F}}$ to these subgroups, to obtain subgroups of $\mathrm{AGL}_2(3)$. Note that $|\mathrm{GL}_2(3)| = 48$ and $|\mathrm{AGL}_2(3)| = 9 \times 42 = 432$.

- First, recall the special linear group $\mathrm{SL}_n(\mathbb{F})$, the subgroup of $\mathrm{GL}_n(\mathbb{F})$ where every element has determinant equal to 1. Here, we are interested in $\mathrm{SL}_2(3) < \mathrm{GL}_2(3)$ which has order 24. Now, restricting our action to just $\mathrm{SL}_2(3)$ acting on V as an object in $\mathfrak{Vect}_{\mathbb{F}}$ we obtain the affine special linear group $\mathrm{ASL}_2(3)$.

- Let $\theta = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ and $\lambda = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. It is easy to see that both of these are order 4 elements in $\mathrm{SL}_2(3)$, hence, $\langle\theta\rangle \cong \langle\lambda\rangle \cong C_4$. Moreover, $\theta^2 = \lambda^2 = (\theta\lambda)^2$. Now, recall the definition of the Quaternion group $Q_8 = \langle i, j, k \colon i^2 = j^2 = k^2 = ijk,\ (ijk)^2 = 1\rangle$. Setting $i = \theta$, $j = \lambda$ and $k = \theta\lambda$ we can see that $\langle\theta, \lambda\rangle \cong Q_8$. We obtain a new subgroup $\mathbb{F}_3^2 \rtimes_\eta Q_8$ of $\mathrm{AGL}_2(3)$. We will later find out this group is the Mathieu group M_9.

- Take $\gamma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, an element of order 3 in $\mathrm{SL}_2(3)$. Restricting the action η to $\langle \gamma \rangle$ we obtain $\mathbb{F}_3^2 \rtimes_\eta C_3$, which is isomorphic to the Heisenberg group over $\mathbb{F}_3$ denoted $\mathrm{He}(3)$. Note that $\mathrm{He}(3)$ is usually defined to be the set of unitriangular 3×3 matrices over $\mathbb{F}_3$.

Finally we note that $\mathrm{SL}_2(3) = \langle \theta, \lambda, \gamma \rangle$ and taking any element $\omega \in \mathrm{GL}_2(3)$ with determinant not equal to 1 then $\mathrm{GL}_2(3) = \langle \theta, \lambda, \delta, \omega \rangle$. Figure 2.1 shows a lattice diagram of the groups we have just constructed.

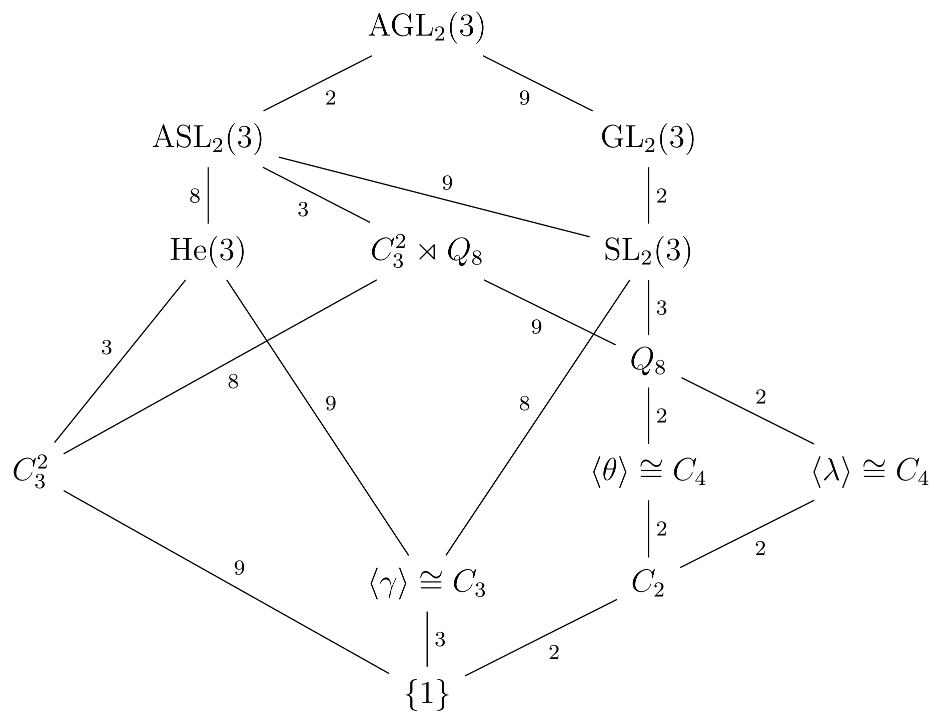


Figure 2.1: Subgroup lattice of $AGL_2(3)$, the numbers represent index.

We note that the four groups at the top left of the diagram are of Affine type. Each of them contain a subgroup C_3^2 which is isomorphic to the vector space $\mathbb{F}_3^2$. In fact the group C_3^2 is also of Affine type. It is technically being acted on trivially by the trivial group; that is $C_3^2 \cong C_3^2 \rtimes_\eta \{1\}$.

Chapter 3

Group Representation Theory

Throughout this chapter, we will develop the notion of *group representation theory*; that is, the study of groups acting on vector spaces. Our treatment will be fairly standard and will follow ideas developed in [Bur93, Isa94, JL01]. The key questions to keep in mind are, if G is a group acting on a vector space V as an object in $\mathfrak{Vect}_{\mathbb{F}}$, then what subspaces of V are left invariant under the action of G? And how many different ways can a group be written as a set of (complex) matrices?

3.1 An Introduction

Definition 3.1. *Representation of a Group.*
A representation of a group G on a vector space V over a field $\mathbb{F}$ is a triple (G, V, ϕ), consisting of G a group, V a vector space and a group homomorphism $\phi : G \to \mathrm{GL}(V)$; that is $\forall g, h \in G$ we have $\phi(gh) = \phi(g)\phi(h)$. Equivalently, we can think of a representation as an action of G on V as an object in $\mathfrak{Vect}_{\mathbb{F}}$.

We should note that ϕ does not need to be a monomorphism, but if it is, the representation is described as being *faithful*; this is exactly the same as the corresponding action of G on V as an object in $\mathfrak{Vect}_{\mathbb{F}}$ being faithful.

Example 13. *Trivial and Regular Representations*
The following representations can be constructed for any finite group G:

(i) The *trivial representation* is simply the trivial homomorphism $\iota : G \to \mathrm{GL}(V)$ by $\iota : g \mapsto 1_V$.

(ii) We can construct the *(left) regular representation* of G by identifying the group elements with a basis of V and taking any $g \in G$. We can then define a left action of G

on the basis of V by $\rho(g,x) \mapsto gx,\ \forall x \in G$. That is if $G = \{g_1, \ldots, g_n\}$ and we identify each g_i with an $\mathbf{e}_i$ from a basis $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for V, we obtain a new basis $B_G = \{\mathbf{e}_{g_1}, \ldots, \mathbf{e}_{g_n}\}$. Then $h \in G$ acts on B_G by $h\mathbf{e}_{g_i} = \mathbf{e}_{h(g_i)}$. The action can be extended linearly, that is, $h \in G$ acts by $h(\sum_{i=1}^n f_i \mathbf{e}_{g_i}) = \sum_{i=1}^n f_i h(\mathbf{e}_{g_i})$ where $f_i \in \mathbb{F}$.

We should note that the (left) regular representation includes the (left) regular action of G (Section 2.3) because when G acts on V, it acts via the basis identified with G. Moreover, the action on the basis is exactly equivalent to the (left) regular action.

Example 14. *Regular Representation of Sym(3)*
We have already seen an example of the regular representation of C_3 in Example 11. Now we will look at a slightly bigger group. Instead of writing out the representation of every element in $Sym(3)$, we will just look at what happens to the generating set $\langle (12), (123) \rangle$.

Firstly, we will write down our basis $B_{Sym(3)} = \{\mathbf{e}_1, \mathbf{e}_{(12)}, \mathbf{e}_{(13)}, \mathbf{e}_{(23)}, \mathbf{e}_{(123)}, \mathbf{e}_{(132)}\}$ for $V \cong \mathbb{C}^6$. Now we need to see where our generators send the other elements in the group under the left regular action; fortunately this is as simple as reading off the Cayley table[1][Cay89].

·	1	(12)	(13)	(23)	(123)	(132)
(12)	(12)	1	(132)	(123)	(23)	(13)
(123)	(123)	(13)	(23)	(12)	(132)	1

Define $\phi : Sym(3) \to \mathrm{GL}(V)$, now we can write down where ϕ maps our generators:

$$\phi((12)) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \phi((123)) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 15. *Permutation Representations*
Consider a permutation group G that acts naturally on a set of size n as an object in $\mathfrak{Set}$. An incredibly useful representation of G can be constructed using the natural action of G. We do this by assigning each of the n points to a basis vector; we can then map every $g \in G$ to a matrix which permutes the basis vectors in the same way that G acts on the set.

This is best illustrated with an example. Take $G = Sym(3)$ and a set $\Delta = \{1, 2, 3\}$. We shall let G act on Δ in the 'natural' way (apply the permutation to an element of the set). Now, define $B_\Delta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$; we can adapt the action by applying the permutation to the number assigned to each basis vector, for example: $(123) \cdot \mathbf{e}_1 = \mathbf{e}_2$. We can now calculate generators for the permutation representation of $Sym(3)$. Define $\phi : Sym(3) \to \mathrm{GL}_3(\mathbb{C})$ by:

[1]Recall that the Cayley table multiplication order is row·column.

$$\phi((12)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \phi((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Now we know what a representation is and how to calculate it, the obvious question we should be asking is: How many representations are there for a given group? Before we can answer this, we should first have a way to tell if two representations are equivalent.

Definition 3.2. *Equivalent Representations*
Given a group G and two vector spaces V and W over $\mathbb{F}$ with corresponding representations ρ and ϕ, then ρ and ϕ are said to be *equivalent* if there exists an isomorphism $\tau : V \to W$, such that $\forall g \in G$, we have $\tau\rho(g)\tau^{-1} = \phi(g)$. If there is such a τ we write $\rho \sim \phi$. This can be summarised by the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle\tau} \\ W & \xrightarrow[\phi(g)]{} & W \end{array}$$

Another equivalent interpretation is that we are checking whether the matrices corresponding to each representation are *simultaneously similar*.

Example 16. *Equivalent and Inequivalent Representations*

1. Define two representations of $\{1, g\}$ on $V = \mathbb{C}^2$. That is, define $\rho, \phi : C_2 \to \mathrm{GL}_2(\mathbb{C})$ by:

$$\rho(1) = \phi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \rho(g) = \tfrac{1}{2}\begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix} \text{ and } \ \phi(g) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is easy to check that ρ and ϕ are well defined group homomorphisms. Now let $\tau : V \to V$ be a vector space isomorphism with $\tau = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$. A quick calculation gives $\tau^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$. Now observe:

$$\tau^{-1}\phi(g)\tau = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix} = \rho(g).$$

As we can conjugate ϕ by τ to obtain ρ, we see that $\rho \sim \phi$.

2. We shall now consider another two representations of C_2, this time on $V \cong (\mathbb{F}_5)^3$. Define $\theta, \psi : C_2 \to \mathrm{GL}_3(5)$ by:

$$\theta(1) = \psi(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \theta(g) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } \psi(g) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Can we find a τ such that $\tau\theta\tau^{-1} = \psi$? What we are really asking here is, is $\theta(g)$ similar to $\psi(g)$? Checking determinants, we see that $\det(\theta(g)) = \det(\psi(g)) = 4$ (as we are working over $\mathbb{F}_5$). We should also check their eigenvalues. A quick calculation gives the eigenvalues as $4, 4, 4$ for $\theta(g)$ and $4, 1, 1$ for $\psi(g)$. Hence, the matrices are not similar and there does not exist such a τ. Therefore $\theta \not\sim \psi$.

3.2 Rings, Modules and the Group Algebra

Clearly, the regular representation is easy to calculate for small groups. However, if we want to look at some of the more interesting phenomena that occur within group theory, we will need to look at bigger groups. Unfortunately writing down matrices with hundreds of rows and columns is certainly not practical. We are clearly going to need a more technical approach to analyse larger groups.

The following section follows the treatment given in chapter 1 of Isaacs's book 'Character Theory of Finite Groups' [Isa94] and chapter 2 of Martin Burrow's book 'Representation Theory of Finite Groups' [Bur93].

Definition 3.3. [Isa94, p. 1] $\mathbb{F}$-*Algebra*
Let $\mathcal{A}$ be a vector space over $\mathbb{F}$ with the structure of a ring with identity. Suppose that for all $f \in \mathbb{F}$ and $a, b \in \mathcal{A}$ that:

$$(fa)b = f(ab) = a(fb). \tag{3.1}$$

Then $\mathcal{A}$ is an $\mathbb{F}$-*Algebra*.

Definition 3.4. [Isa94, p. 3] $\mathbb{F}$-*Algebra Homomorphism*
Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbb{F}$-Algebras, then $\forall x, y \in \mathcal{A}$ an $\mathbb{F}$-*Algebra homomorphism* $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies:

(1) $\Phi(xy) = \Phi(x)\Phi(y)$ and moreover $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$;

(2) Φ is an $\mathbb{F}$-linear transformation.

An $\mathbb{F}$-*Algebra Isomorphism* is an $\mathbb{F}$-*Algebra Homomorphism* that is also invertible.

We can now define the category $\mathfrak{Alg}_{\mathbb{F}}$ where the objects are $\mathbb{F}$-Algebras and the arrows are $\mathbb{F}$-Algebra homomorphisms.

Example 17. $\mathbb{F}$-*Algebras*

A simple example of an $\mathbb{F}$-Algebra is $\mathbf{M}_2(\mathbb{R})$; here our vector space is $\mathbb{R}^4$ but we shall write elements as 2×2 matrices. The ring structure is given by considering standard matrix multiplication as the ring's multiplication. As scalars commute with matrices, verifying (3.1) is straightforward:

$$(fa)b = \left(f \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} f\right) \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \left(f \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\right)$$

$$= a(fb) = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} fb_1 & fb_2 \\ fb_3 & fb_4 \end{bmatrix} = \begin{bmatrix} fa_1b_1 + fa_2b_3 & fa_1b_2 + fa_2b_4 \\ fa_3b_1 + fa_4b_3 & fa_3b_2 + fa_4b_4 \end{bmatrix} = f(ab).$$

Another example is the *Group Algebra* denoted $\mathbb{F}[G]$; this will be of primary use to us. To construct the group algebra for a group G with order n over a field $\mathbb{F}$, we must first start with an n dimensional vector space over $\mathbb{F}$. We should assign a basis B to our vector space such that the elements of the basis are labeled after the group elements, that is $B = \{g_1, \ldots, g_n\}$. Now each element a of $\mathbb{F}[G]$ can be expressed as a formal sum:

$$a = \sum_{i=1}^{n} f_i g_i \text{ where } f_i \in \mathbb{F}. \tag{3.2}$$

We define multiplication on $\mathbb{F}[G]$ as the multiplication of the basis vectors with respect to the group operation and extend linearly. For $a, b \in \mathcal{A}$ we have

$$a \cdot b = \sum_{i=1}^{n} a_i g_i \sum_{j=1}^{n} b_j g_j = \sum_{i,j=1}^{n} a_i b_j g_i g_j \text{ where } a_i, b_i \in \mathbb{F}.$$

We should note that (3.2) is actually a linear combination of basis vectors. Moreover, the identification with G means we have an instant embedding of G into $\mathbb{F}[G]$.

We can now use the structure of the group algebra to study group representations. In fact, we shall take a look at the action of the group algebra $\mathbb{F}[G]$ on a vector space V in $\mathfrak{Vect}_{\mathbb{F}}$. We can define a (left) action ψ of a group algebra on a vector space $V \in \mathfrak{Vect}_{\mathbb{F}}$ analogously to that of a group; it is a function of the form:

$$\Psi : \mathbb{F}[G] \times V \to V, \quad \Psi(a, \mathbf{v}) \mapsto a \cdot \mathbf{v}.$$

We require the action satisfies five properties; the first two are exactly the properties of a group action (Definition 2.5) and the other three are as follows:

- $\forall a, b \in \mathcal{A},\ \mathbf{v} \in V$ we have $(a+b)\mathbf{v} = a \cdot \mathbf{v} + b \cdot \mathbf{v}$;
- $\forall a \in \mathcal{A},\ \mathbf{v}, \mathbf{w} \in V$ we have $a \cdot (\mathbf{v} + \mathbf{w}) = a \cdot \mathbf{v} + a \cdot \mathbf{w}$;
- $\forall a \in \mathcal{A},\ \mathbf{v} \in V,\ f \in \mathbb{F}$ we have $a \cdot (f\mathbf{v}) = f(a \cdot \mathbf{v}) = (fa) \cdot \mathbf{v}$.

In much the same way that an action of a group on V as an object in $\mathfrak{Vect}_{\mathbb{F}}$ gave a homomorphism from G to $\mathrm{GL}(V)$, the action of $\mathbb{F}[G]$ on V as an object in $\mathfrak{Vect}_{\mathbb{F}}$ gives a homomorphism from $\mathbb{F}[G]$ to $\mathrm{End}(V) = \mathrm{Hom}_{\mathfrak{Vect}_{\mathbb{F}}}[V, V]$, the endomorphism algebra of V. Recall that $\mathrm{End}(V) \cong \mathbf{M}_n(\mathbb{F})$ if V is finite dimensional with dimension n.

Lemma 3.5. *Every action of $\mathbb{F}[G]$ on V as an object in $\mathfrak{Vect}_{\mathbb{F}}$ can be restricted to an action of G on V as an object in $\mathfrak{Vect}_{\mathbb{F}}$. Conversely, every action of G on V as an object in $\mathfrak{Vect}_{\mathbb{F}}$ can be extended to an action of $\mathbb{F}[G]$ on V as an object in $\mathfrak{Vect}_{\mathbb{F}}$.*

Proof. If we start with an action Ψ of $\mathbb{F}[G]$ on V as an object in $\mathfrak{Vect}_{\mathbb{F}}$, then by definition the restriction ψ is a group action of G on V as an object in $\mathfrak{Vect}_{\mathbb{F}}$. Conversely, if we start with an action ψ of G on V as an object in $\mathfrak{Vect}_{\mathbb{F}}$, then construct our group algebra $\mathbb{F}[G]$ in the standard way (3.2); we can extend the action to Ψ as follows:

$$\Psi\left(\sum_{i=1}^{n} f_i g_i, \mathbf{v}\right) \mapsto \sum_{i=1}^{n} f_i g_i(\mathbf{v}).$$

Now observe that $\Psi(1, \mathbf{v}) = \psi(1, \mathbf{v}) = \mathbf{v}$ and for all $a, b \in \mathcal{A}$ we have:

$$\sum_{i=1}^{n} a_i g_i \left(\sum_{j=1}^{n} b_j g_j \cdot \mathbf{v}\right) = \left(\sum_{i=1}^{n} a_i g_i \sum_{j=1}^{n} b_j g_j\right) \cdot \mathbf{v} = \sum_{i,j=1}^{n} a_i b_j g_i g_j(\mathbf{v}).$$

For the first additive property we have:

$$\left(\sum_{i=1}^{n} a_i g_i + \sum_{i=1}^{n} b_i g_i\right) \cdot \mathbf{v} = \left(\sum_{i=1}^{n} (a_i + b_i) g_i\right) \cdot \mathbf{v} = \sum_{i=1}^{n} (a_i + b_i) g_i(\mathbf{v}) = \sum_{i=1}^{n} a_i g_i(\mathbf{v}) + \sum_{i=1}^{n} b_i g_i(\mathbf{v}).$$

For the second:

$$\sum_{i=1}^{n} a_i g_i(\mathbf{v} + \mathbf{w}) = \sum_{i=1}^{n} a_i g_i \cdot (\mathbf{v} + \mathbf{w}) = \left(\sum_{i=1}^{n} a_i g_i\right) \cdot \mathbf{v} + \left(\sum_{i=1}^{n} a_i g_i\right) \cdot \mathbf{w} = \sum_{i=1}^{n} a_i g_i(\mathbf{v}) + \sum_{i=1}^{n} a_i g_i(\mathbf{w}).$$

Scalar multiplication is immediate:

$$\sum_{i=1}^{n} a_i g_i \cdot (f\mathbf{v}) = f\left(\sum_{i=1}^{n} a_i g_i \cdot \mathbf{v}\right) = f\left(\sum_{i=1}^{n} a_i g_i(\mathbf{v})\right) = \sum_{i=1}^{n} f a_i g_i(\mathbf{v}) = \sum_{i=1}^{n} f a_i g_i \cdot \mathbf{v}.$$

□

Definition 3.6. [Isa94, p. 3] *(Left) $\mathcal{A}$-Module*
Let $\mathcal{A}$ be an $\mathbb{F}$-Algebra and M be a finite dimensional vector space over $\mathbb{F}$. For every $a \in \mathcal{A}$ and $\mathbf{m} \in M$, define $\mathbf{m}a$ to be some element of M. Now, assume for all $a, b \in \mathcal{A}$, $\mathbf{m}, \mathbf{n} \in M$ and $f \in \mathbb{F}$ that:

(M1) $a(\mathbf{m} + \mathbf{n}) = a\mathbf{m} + a\mathbf{n}$.

(M2) $(a + b)\mathbf{m} = a\mathbf{m} + b\mathbf{m}$.

(M3) $(ab)\mathbf{m} = a(b\mathbf{m})$.

(M4) $a(f\mathbf{m}) = f(a\mathbf{m}) = (fa)\mathbf{m}$.

(M5) $1\mathbf{m} = \mathbf{m}$.

Then M is a *(Left) $\mathcal{A}$-Module.*

From this point on, all $\mathcal{A}$-Modules will be (left) $\mathcal{A}$-Modules and we will drop the prefix (left). The definition of an $\mathcal{A}$-Module is another way of looking at the action of $\mathcal{A}$ on an object in $\mathfrak{Vect}_{\mathbb{F}}$. In fact, if M is an $\mathcal{A}$-Module, then $\mathcal{A}$ acts on M as an object in $\mathfrak{Vect}_{\mathbb{F}}$. From this we can see that the Modules of a group algebra $\mathbb{F}[G]$ correspond to the representations of the group G.

Example 18. *Examples of $\mathcal{A}$-Modules*

- If $\mathcal{A}$ is any $\mathbb{F}$-Algebra then $\mathcal{A}$ is an $\mathcal{A}$-Module under left multiplication. We will call this the *regular* $\mathcal{A}$-Module and denote it $\hat{\mathcal{A}}$.
- Take $\mathbb{F}$ as an $\mathbb{F}$-Algebra, then an $\mathbb{F}$-Module is exactly a vector space over $\mathbb{F}$; in fact, vector spaces are exactly modules over fields.

Definition 3.7. [Isa94, p. 3] *$\mathcal{A}$-Module Homomorphism*
Let M and N be $\mathcal{A}$-Modules and define a linear transformation $\Phi : M \to N$. Then Φ is an *$\mathcal{A}$-Module homomorphism* if for all $\mathbf{m} \in M$ and $a \in \mathcal{A}$ we have: $\Phi(a\mathbf{m}) = a\Phi(\mathbf{m})$. An $\mathcal{A}$-Module homomorphism is an *$\mathcal{A}$-Module isomorphism* if it is invertible.

We make the following remark in regards to $\mathcal{A}$-Modules. The modules over an $\mathbb{F}$-Algebra $\mathcal{A}$ form a category $\mathfrak{Mod}_{\mathcal{A}}$; we will denote the Hom-space of any two modules M and N in this category by $\mathrm{Hom}_{\mathcal{A}}[M, N]$.

Lemma 3.8. *If two $\mathbb{F}[G]$-Modules M and N are isomorphic, then their corresponding representations are equivalent.*

Proof. Let $M \cong N$ be two $\mathbb{F}[G]$-Modules, and let μ and η be their corresponding representations. As $M \cong N$, we have an $\mathbb{F}[G]$-Module isomorphism $\mathcal{T} : M \to N$, recall that $\mathcal{T}(a\mathbf{m}) = a\mathcal{T}(\mathbf{m})$ for all $a \in \mathbb{F}[G]$ and $\mathbf{m} \in M$. Now let B_M and B_N be basis for M and N respectively; by definition, $\mathcal{T}$ is linear, so $\mathcal{T}$ has an invertible matrix representation τ with respect to B_M and B_N. It immediately follows that $\mu(a)\tau = \tau\eta(a)$ for all $a \in \mathbb{F}[G]$ and therefore $\mu \sim \eta$. □

We remark that the converse of the previous lemma is also true. We have discussed several ideas throughout this chapter; we will summarise these by stating that the following ideas are equivalent:

1. Representations of a group G.
2. Group actions of G on a vector space V in $\mathfrak{Vect}_{\mathbb{F}}$.
3. Actions of the group algebra $\mathbb{F}[G]$ on a vector space V in $\mathfrak{Vect}_{\mathbb{F}}$.
4. $\mathbb{F}[G]$-modules.

The notion of equivalence here is formal; each class of objects with the arrows between them forms a category. These categories are all equivalent to $\mathfrak{Mod}_{\mathbb{F}[G]}$.

3.3 Submodules

Definition 3.9. [Isa94, p. 3] *Submodules*
Given an $\mathcal{A}$-module M, then a submodule N is a subspace of M such that $\forall a \in \mathcal{A}$ and $\forall \mathbf{n} \in N$ we have $a\mathbf{n} \in N$. A nonzero $\mathcal{A}$-Module M is termed *irreducible* if its only submodules are $\{0\}$ and M. Moreover, the representation corresponding to an $\mathcal{A}$-module is termed *irreducible* if and only if the corresponding $\mathcal{A}$-module is *irreducible.*

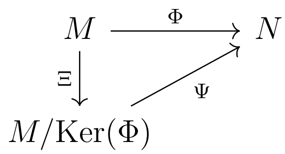
Definition 3.10. [JL01, p. 66] *Direct Sum*
Let M and N be $\mathbb{F}[G]$-modules with respective basis $B_1 = \{\mathbf{m}_1, \ldots, \mathbf{m}_k\}$ and $B_2 = \{\mathbf{n}_1, \ldots, \mathbf{n}_l\}$. We define the *direct sum* of M and N, denoted $M \oplus N$, as the $\mathbb{F}[G]$-module with basis $B = \{\mathbf{m}_1, \ldots, \mathbf{m}_k, \mathbf{n}_1, \ldots \mathbf{n}_l\}$. If $g \in G$ then,

$$[g]_B = \left[\begin{array}{c|c} [g]_{B_1} & 0 \\ \hline 0 & [g]_{B_2} \end{array}\right].$$

Theorem 3.11. [JL01, p. 61] The Fundamental Theorem of Module Homomorphisms
Let M and N be $\mathbb{F}[G]$-submodules and $\Phi : M \to N$ an $\mathbb{F}[G]$-homomorphism. Then:

1. $\mathrm{Ker}(\Phi)$ *is a submodule of* M*;*
2. $\mathrm{Im}(\Phi)$ *is a submodule of* N*;*
3. $\mathrm{Im}(\Phi) \cong M/\mathrm{Ker}(\Phi)$.

This theorem can be summarised by a commutative diagram. Here, Ξ is the canonical projection from M to $M/\mathrm{Ker}(\Phi)$ and Ψ is an isomorphism between $M/\mathrm{Ker}(\Phi)$ and $\mathrm{Im}(\Phi)$.



A proof of this result can be found in James and Liebeck's 'Representations and Characters of Groups' [JL01, p. 61].

Lemma 3.12. [JL01, p. 67] *Let* M *be an* $\mathbb{F}[G]$*-module and suppose that* $M = M_1 \oplus \cdots \oplus M_k$*, so for* $\mathbf{m} \in M$ *we have* $\mathbf{m} = \mathbf{m}_1 + \cdots + \mathbf{m}_k$ *with* $\mathbf{m}_i \in M_i$*. Define* $\pi_i \colon M \to M$ *by* $\pi_i(\mathbf{m}) = \mathbf{m}_i$*; then* π_i *is an* $\mathbb{F}[G]$*-homomorphism whose image is isomorphic to* M_i*.*

Proof. First, we shall verify the properties of an $\mathbb{F}[G]$-homomorphism. Clearly, π_i is an $\mathbb{F}$-linear transformation. Now, for $\mathbf{m} \in M$ and $g \in G$ we have:

$$\pi_i(g\mathbf{m}) = \pi_i(g\mathbf{m}_1 + \cdots + g\mathbf{m}_k) = g\mathbf{m}_i = g\pi_i(\mathbf{m}).$$

Hence, π_i is an $\mathbb{F}[G]$-homomorphism. The fact that $\mathrm{Im}(\pi_i) = M_i$ is immediate from the definition of π_i. □

Example 19. *Submodules of the Regular Representation of* $Sym(3)$
In Example 14, we calculated the regular representation of $Sym(3)$ on $V \cong \mathbb{C}^6$ with basis $B_{Sym(3)} = \{\mathbf{e}_1, \mathbf{e}_{(12)}, \mathbf{e}_{(13)}, \mathbf{e}_{(23)}, \mathbf{e}_{(123)}, \mathbf{e}_{(132)}\}$; from another point of view, V is the regular $\mathbb{C}[Sym(3)]$-Module.

If we let $\mathbf{x} = \mathbf{e}_1 + \mathbf{e}_{(12)} + \mathbf{e}_{(13)} + \mathbf{e}_{(23)} + \mathbf{e}_{(123)} + \mathbf{e}_{(132)}$, then it is easy to check that the subspace generated by $\mathbf{x}$, denoted $\langle \mathbf{x} \rangle$, is invariant under $\mathbb{C}[Sym(3)]$. Moreover, this submodule is actually the trivial representation of $Sym(3)$. Note that $\langle \mathbf{x} \rangle$ is of dimension 1, making it irreducible.

Recall the Sgn homomorphism of $Sym(n)$ we used in Example 5. We can use this to build a representation of $Sym(3)$. As Sgn sent even permutations to 1 and odd permutations to -1, let $\mathbf{Sgn} = \mathbf{e}_1 - \mathbf{e}_{(12)} - \mathbf{e}_{(13)} - \mathbf{e}_{(23)} + \mathbf{e}_{(123)} + \mathbf{e}_{(132)}$, then the submodule $\langle \mathbf{Sgn} \rangle$ is one dimensional and as such is irreducible.

It is easy to see that $\langle \mathbf{x} \rangle$ and $\langle \mathbf{Sgn} \rangle$ are not equivalent. The eigenvalue for every group element is 1 under the trivial representation, whereas for the Sgn representation the odd permutations have an eigenvalue of -1. Clearly, we cannot define a linear transform to map one to the other. To calculate any more submodules we will need some more machinery.

Definition 3.13. *Completely Reducible Modules*
We describe M as *completely reducible* if it can be expressed as the direct sum of irreducible modules; that is $M = \bigoplus_i M_i$ where M_i is an irreducible submodule of M. Moreover, *completely reducible* implies there exist short exact sequences that split[2] (Definition 2.4) as follows:

$$0 \longrightarrow M_i \xrightarrow{f_1} M \xrightarrow{f_2} \frac{M}{M_i} \longrightarrow 0$$

If every $\mathcal{A}$-module is *completely reducible* then we describe $\mathcal{A}$ as *semi-simple*. We should note that irreducible modules are by definition, completely reducible.

We will soon see that we can break down $\widehat{\mathbb{F}[G]}$ into a set of unique irreducible modules. Correspondingly, we will get a set of irreducible representations that are unique up to equivalence.

Lemma 3.14. [Sch04] Schur's Lemma
If M and N are irreducible $\mathcal{A}$-Modules, then every nonzero element of $\mathrm{Hom}_{\mathcal{A}}[M, N]$ has an inverse in $\mathrm{Hom}_{\mathcal{A}}[N, M]$

Proof. Let $\Phi \in \mathrm{Hom}_{\mathcal{A}}[M, N]$ then $\mathrm{Ker}(\Phi)$ is a submodule of M and $\mathrm{Im}(\Phi)$ is a submodule of N. The Lemma is now immediate. □

Corollary 3.15. [JL01, p. 78] *Let M and N be irreducible $\mathcal{A}$-Modules and let $\Phi : M \to N$ be an $\mathcal{A}$-Module homomorphism, then either $M \cong N$ or Φ is the trivial homomorphism.*

Corollary 3.16. [JL01, p. 78] *Let M be irreducible and $\Phi : M \to M$ an isomorphism. Then Φ is equal to scalar multiplication by some $f \in \mathbb{F}$.*

3.4 Maschke's Theorem

Theorem 3.17. Maschke's Theorem
Let G be a finite group of order n and $\mathbb{F}$ a field with characteristic p, then every $\mathbb{F}[G]$-Module is completely reducible if and only if $p = 0$ or $p \nmid n$.

There are a number of proofs for Maschke's Theorem, however, many of these require studying the Jacobson radical of a Ring and Nilpotent ideals. Studying these would increase

[2] As modules are constructed over Abelian groups, a quotient module is constructed in exactly the same way as a quotient group with the additional condition that it has the structure of a module.

the length of this project significantly and so the interested reader is referred to Chapter 2 of Burrow's book 'Representation Theory of Finite Groups' [Bur93]. If the reader has some knowledge of the German language, then they are also referred to Maschke's original articles[Mas98][Mas99].

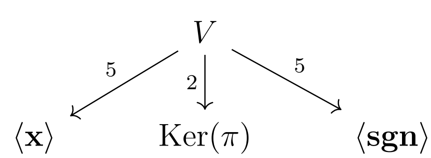
We will now restrict to fields of characteristic zero, specifically, the field of complex numbers (that is $\mathbb{F} = \mathbb{C}$). By restricting to $\mathbb{C}$, Maschke's Theorem holds and then every group algebra will be *semi-simple*. We now have a framework for calculating and breaking down group representations (up to equivalence).

Lemma 3.18. [Isa94, p. 7] *Every irreducible $\mathbb{C}[G]$-module M is isomorphic to a submodule of the regular module $\widehat{\mathbb{C}[G]}$.*

Proof. Let M be an irreducible $\mathbb{C}[G]$-module and choose $\mathbf{m} \in M$ such that $\mathbf{m} \neq 0$. Define a projection $\Phi : \widehat{\mathbb{C}[G]} \to M$ by $\Phi(a) = a\mathbf{m}$. As $\Phi(ab) = ab\mathbf{m} = a\Phi(b)$, then Φ is a $\mathbb{C}[G]$-module homomorphism. Now $\mathbf{m} \in \mathrm{Im}(\Phi)$ and M is irreducible, so by Schur's Lemma (Lemma 3.14), $\mathrm{Im}(\Phi) = M$. Hence, $M \cong \frac{\widehat{\mathbb{C}[G]}}{\mathrm{Ker}(\Phi)}$ and by Maschke's Theorem $\widehat{\mathbb{C}[G]} \cong M \oplus \mathrm{Ker}(\Phi)$. □

The immediate consequence of Lemma 3.18 is that we can now hope to explicitly write down every representation of group G over $\mathbb{C}$ up to equivalence.

Example 20. *Revisiting the Submodules of the Regular Representation of $Sym(3)$*
Recall that we calculated two irreducible submodules of $Sym(3)$, $\langle \mathbf{x} \rangle$ and $\langle \mathbf{Sgn} \rangle$. Let $H = \langle \mathbf{x} \rangle \oplus \langle \mathbf{Sgn} \rangle$ and define $\pi : V \to H$. If there are any more irreducibles they will be contained within $\mathrm{Ker}(\pi)$, however, at this point we should summarise the situation with a diagram of containments (the numbers represent codimension):



Let $\omega = e^{\frac{2i\pi}{3}}$ and define:

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{e}_1 + \omega\mathbf{e}_{(123)} + \omega^2\mathbf{e}_{(132)} \\
\mathbf{k}_2 &= \mathbf{e}_1 + \omega^2\mathbf{e}_{(123)} + \omega\mathbf{e}_{(132)} \\
\mathbf{k}_3 &= \mathbf{e}_{(12)} + \omega\mathbf{e}_{(23)} + \omega^2\mathbf{e}_{(13)} \\
\mathbf{k}_4 &= \mathbf{e}_{(12)} + \omega^2\mathbf{e}_{(23)} + \omega\mathbf{e}_{(13)}
\end{aligned}$$

We claim that $\mathrm{Ker}(\pi) = \langle \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 \rangle$. To see this, note that the $\mathbf{k}_i$ are linearly independent and that the set is fixed under the action of $Sym(3)$. Furthermore, each of the $\mathbf{k}_i$ is clearly in $\mathrm{Ker}(\pi)$; thus, the fact that $\dim(\mathrm{Ker}(\pi)) = 4$ yields the claim.

The action of $Sym(3)$ on $\mathrm{Ker}(\pi)$ has two orbits $\{\mathbf{k}_2, \mathbf{k}_4\}$ and $\{\mathbf{k}_1, \mathbf{k}_3\}$. Therefore $K_1 = \langle \mathbf{k}_1, \mathbf{k}_3 \rangle$ and $K_2 = \langle \mathbf{k}_2, \mathbf{k}_4 \rangle$ are 2-dimensional submodules of $\mathbb{C}[Sym(3)]$. Define the function:

$$\Psi \colon K_1 \to K_2 \text{ by } \mathbf{k}_1 \mapsto \mathbf{k}_2,\ \mathbf{k}_3 \mapsto \mathbf{k}_4.$$

Then, Ψ is a $\mathbb{C}[Sym(3)]$-module isomorphism.

Theorem 3.19. *There are only finitely many non isomorphic irreducible $\mathbb{C}[G]$-Modules M_i and the sum of their dimensions is less than or equal to the order of G.*

Proof. Let $\widehat{\mathbb{C}[G]} \cong \bigoplus_i M_i$ where M_i is irreducible and let M be any irreducible $\mathbb{C}[G]$-module. There exists a projection $\Phi : \widehat{\mathbb{C}[G]} \to M_i$ that does not map M to zero (see proof of Lemma 3.18). Now, restrict Φ to $\Phi : M \to M_i$. There must be a choice for i for which this is non zero. By Schur's Lemma (Lemma 3.14) we have a module isomorphism.

Now, $M_1, \ldots, M_k$ contains all irreducible $\mathbb{C}[G]$-modules. Observe that $\dim(M_1) + \cdots + \dim(M_k) = \dim(\widehat{\mathbb{C}[G]}) = |G|$. However, some of these $\mathbb{C}[G]$-modules may be isomorphic, so the sum of their dimensions is less than or equal to $|G|$. □

Lemma 3.20. [JL01, p. 97] *For any $\mathbb{C}[G]$-Module M,* $\dim\left(\mathrm{Hom}_{\mathbb{C}[G]}\left[\widehat{\mathbb{C}[G]}, M\right]\right) = \dim(M)$

Proof. (Sketch). Let $\dim(M) = n$ and let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for M. Define $\Phi_i : \widehat{\mathbb{C}[G]}$ by $a \mapsto a\mathbf{e}_i$, then $\Phi_i \in \mathrm{Hom}_{\mathbb{C}[G]}\left[\widehat{\mathbb{C}[G]}, M\right]$. Moreover, $\{\Phi_1, \ldots \Phi_n\}$ is a basis for $\mathrm{Hom}_{\mathbb{C}[G]}\left[\widehat{\mathbb{C}[G]}, M\right]$. □

Theorem 3.21. [JL01, p. 101] *Let $\mathtt{M}$ be the set of unique irreducible $\mathbb{C}[G]$-Modules, then:*

$$|G| = \sum_{M \in \mathtt{M}} \dim(M)^2.$$

Proof. Now, $|G| = \dim\left(\widehat{\mathbb{C}[G]}\right) = \sum \dim(M)$ where M is any irreducible $\mathbb{C}[G]$-Module. By Lemma 3.20 each $M \in \mathtt{M}$ must occur in $\widehat{\mathbb{C}[G]}$, $\dim(M)$ times; the result is now immediate. □

Corollary 3.22. *Let, $\mathtt{M}$ be the set of unique irreducible $\mathbb{C}[G]$-Modules and let $k_i = \dim(M_i)$ then*

$$\mathbb{C}[G] = \bigoplus_{M_i \in \mathtt{M}} k_i M_i.$$

In summary, when studying representations of a group G over $\mathbb{C}$, to classify every representation of G, it is enough to calculate all of the unique irreducible $\mathbb{C}[G]$-Modules. The irreducible $\mathbb{C}[G]$-Modules correspond exactly to the irreducible representations of G over $\mathbb{C}$;

the sum of the squares of the dimensions of the unique irreducible $\mathbb{C}[G]$-modules equals the order of the group. Finally, we can reconstruct any possible representation of G by direct summing combinations of the irreducible modules.

Chapter 4

Character Theory

In Chapter 3 we studied several different concepts: Actions on vector spaces, representations and $\mathbb{C}[G]$-modules. We showed that each of these are equivalent ways of studying the same idea. We are now going to develop a method known as *character theory*, which will provide techniques to study and characterise the representations of a finite group G.

The motivation for such a method comes from the fact that representations of larger groups become particularly difficult to calculate and work with. However, with character theory we will be able to completely tabulate the information regarding the characters of a group. We call this table, the *group character table* and not only does it completely characterise the representations of G, but it also gives a large amount of information about the group itself. Just to reiterate, all representations will be over $\mathbb{C}$.

4.1 The Characters of a Group

Definition 4.1. [JL01, p. 117] *Trace of a Matrix*
The *trace* of an $n \times n$ matrix A is defined as the sum of the elements on the diagonal, that is

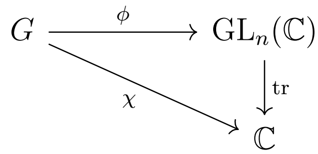
$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{i,i}.$$

Lemma 4.2. [JL01, p. 118] *Let A and B be two $n \times n$ matrices then* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Proof. $\operatorname{tr}(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} b_{j,i} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{j,i} a_{i,j} = \operatorname{tr}(BA)$. □

Definition 4.3. [JL01, p. 119] *Character of a Representation*
Let ϕ be a representation of a group G. The *character* χ_ϕ of G afforded by ϕ is the function

$\chi_\phi(g) = \mathrm{tr}(\phi(g))$. This can be summarised by the following commutative diagram:



If the representation is *irreducible* then we say that the character is *irreducible*. We define $\chi_\phi(1)$ as the *degree* or *dimension* of χ_ϕ. Degree 1 characters are called *linear*.

Example 21. *Some Characters of C_2*
In Example 16 we wrote down some representations of C_2:

$$\rho(1) = \phi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \rho(g) = \tfrac{1}{2}\begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix} \text{ and } \phi(g) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let χ_ρ be the character of ρ and χ_ϕ be the character of ϕ. Then:

$$\chi_\rho(1) = \chi_\phi(1) = \mathrm{tr}(\rho(1)) = 2,\ \chi_\rho(g) = \mathrm{tr}(\rho(g)) = 0 \text{ and } \chi_\phi(g) = \mathrm{tr}(\phi(g)) = 0.$$

Lemma 4.4. [JL01, p. 123] *Let χ be the character of a $\mathbb{C}[G]$-Module M afforded by a representation ϕ then $\chi(1) = \dim(M)$.*

Proof. Let $\dim(M) = d$ then $\phi(1) = I_d$ the $d \times d$ identity matrix, hence $\chi(1) = \mathrm{tr}(I_d) = d$. □

Lemma 4.5. [JL01, p. 119] *Similar representations of a group G afford equal characters and characters are constant on the conjugacy classes of G.*

Proof. Let ϕ and ψ be equivalent representations of G, that is there exists a linear transform τ such that $\tau^{-1}\phi(g)\tau = \psi(g)$. Now fix a basis so we can write τ as a matrix then $\mathrm{tr}(\psi(g)) = \mathrm{tr}(\tau^{-1}\phi(g)\tau) = \mathrm{tr}(\tau\tau^{-1}\phi(g)) = \mathrm{tr}(\phi(g))$. Let $g, h \in G$ then hgh^{-1} is in the conjugacy class of g, now $\mathrm{tr}(\phi(hgh^{-1})) = \mathrm{tr}(\phi(h)\phi(g)\phi(h^{-1})) = \mathrm{tr}(\phi(h)\phi(h^{-1})\phi(g)) = \mathrm{tr}(\phi(g))$. □

There are some important remarks to make, firstly characters characterize representations; by this we mean that much of the information about a representation can be deduced from its character and we will find it is far easier to calculate characters than representations. Secondly to specify a character of group, we only need to specify its value on one element in each conjugacy class, we will usually refer to this element as a *representative element*.

Let $\phi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ be representations of G and define:

$$\theta : G \to \mathrm{GL}(V \oplus W) \text{ by } \theta(g) = \begin{bmatrix} \phi(g) & 0 \\ 0 & \psi(g) \end{bmatrix}.$$

Hence, $\mathrm{tr}(\theta(g)) = \mathrm{tr}(\phi(g)) + \mathrm{tr}(\psi(g))$. Evidently the set of all characters of a group G are closed under addition.

4.2 Counting Representations

Definition 4.6. [Bur93, p. 64] *The Center of a Group Algebra*
The *Center of a Group Algebra* is given by:

$$\mathbf{Z}(\mathbb{C}[G]) = \{z \in \mathbb{C}[G] \colon az = za \ \forall a \in \mathbb{C}[G]\}.$$

This is simply the set of elements which commute with every element in $\mathbb{C}[G]$. We should note that the *Center of a Group Algebra* forms a subalgebra of $\mathbb{C}[G]$.

Lemma 4.7. [Isa94, p. 9] *Given any $z \in \mathbf{Z}(\mathbb{C}[G])$ and any irreducible $\mathbb{C}[G]$-Module M, define the map $\Phi_z \colon \mathbf{m} \mapsto z \cdot \mathbf{m}$; then Φ_z is an M-endomorphism and $\Phi_z(\mathbf{m}) = c\mathbf{m}$ for some $c \in \mathbb{C}$.*

Proof. For every $a \in \mathbb{C}[G]$ we have $az\mathbf{m} = za\mathbf{m}$ hence Φ_z is a $\mathbb{C}[G]$-Module homomorphism with equal source and target. By Schur's Lemma (Lemma 3.14) Φ is an isomorphism and it follows that Φ_z corresponds to scalar multiplication (Corollary 3.16). □

Lemma 4.8. [Isa94, p. 9] *The number of unique irreducible $\mathbb{C}[G]$-Modules is exactly equal to $\dim(\mathbf{Z}(\mathbb{C}[G]))$.*

Proof. We decompose $\mathbb{C}[G]$ as $\bigoplus_i k_i M_i$ where M_i is irreducible and $k_i = \dim(M_i)$. Now let $1 = \sum_i \mathbf{e}_i$ where $\mathbf{e}_i \in k_i M_i$ and let $z \in \mathbf{Z}(\mathbb{C}[G])$; then $z = z1 = \sum_i z(\mathbf{e}_i) = \sum_i f_i \mathbf{e}_i$ where $f_i \in \mathbb{C}$. We see that $\mathbf{Z}(\mathbb{C}[G])$ is a subset of the span of the set of $\mathbf{e}_i$. Any element of the form $\sum_i f_i \mathbf{e}_i$ is in $\mathbf{Z}(\mathbb{C}[G])$, so we see that the $\mathbf{e}_i$ form a basis for $\mathbf{Z}(\mathbb{C}[G])$. There is only one $\mathbf{e}_i$ for each unique irreducible module, hence, $\dim(\mathbf{Z}(\mathbb{C}[G]))$ is equal to the number of unique irreducible $\mathbb{C}[G]$-Modules. □

Lemma 4.9. [Isa94, p. 15] *Let $Cl(1), \ldots, Cl(d)$ be the conjugacy classes of G and let $K_i = \sum_{g \in Cl(i)} g$ then $\{K_1, \ldots, K_d\}$ forms a basis for $\mathbf{Z}(\mathbb{C}[G])$.*

Proof. Let $z \in \mathbf{Z}(\mathbb{C}[G])$ such that $z = \sum_{i=1}^n f_i g_i$ then $\forall h \in G$ we have:

$$z = hzh^{-1} = \sum_{i=1}^{n} f_i g_i = \sum_{j=1}^{n} f_j h g_j h^{-1}.$$

Equating coefficients we see $f_i = f_j$ and $g_i = hg_ih^{-1}$, from this we see that conjugate elements have the same coefficient. Now, define $K_i = \sum_{g \in Cl(i)} g$ then the K_i are linearly independent. It follows that, $z = \sum_{i=1}^d f_i K_i$, thus, the K_i span $\mathbf{Z}(\mathbb{C}[G])$. □

Theorem 4.10. [Bur93, p. 65] *The number of distinct irreducible representations of a finite* *group G is equal to the number of conjugacy classes of G and moreover:*

$$|G| = \sum_{\chi_i \in \mathrm{Irr}(G)} \chi_i(1)^2. \tag{4.1}$$

Where $\mathrm{Irr}(G)$ *is the set of distinct irreducible characters of G.*

Proof. Apply Lemma 4.8 and Lemma 4.9. Moreover, as $\chi_i(1) = \dim(M_i)$ then by Theorem 3.21 we arrive at (4.1). □

Corollary 4.11. [Bur93, p. 67] *Let G be a finite abelian group then every irreducible $\mathbb{C}[G]$-module is of dimension 1.*

Proof. As every element in G is in a singleton conjugacy class, it follows that there are exactly $|G|$ distinct irreducible representations. Moreover, $\sum_{i=1}^{|G|} \chi_i(1)^2 = |G|$ hence $\chi_i(1) = 1$ for every i in $1 \leq i \leq |G|$. □

Theorem 4.12. [JL01, p. 123] *Let G be a finite group and M a $\mathbb{C}[G]$-module. Let $g \in G$ and let g have order n, then there is a basis B of M such that the $[g]_B$ is diagonal. Moreover, the diagonal entries are nth roots of unity.*

Proof. Let $H = \langle g \rangle$, the cyclic subgroup of G generated by g. As M is also a $\mathbb{C}[H]$-module the result is an immediate consequence of Corollary 4.11. □

4.3 Examples of Characters

In this section we will detail several important characters for various representations and some quick ways to calculate them. This is not an exhaustive collection, however, many of these will be useful later on.

4.3.1 Trivial, Regular and Conjugate Characters

For any group G we have the trivial representation $\iota\colon G \to \mathbb{C}$ by $g \mapsto 1$. We denote the *trivial character* by χ_0 and $\chi_0(g) = 1$ for every $g \in G$. The *regular character* is the character afforded by the regular module $\widehat{\mathbb{C}[G]}$ of a group G, denoted χ_G.

Lemma 4.13. [JL01, p. 127] *For $1 \in G$ we have $\chi_G(1) = |G|$ and for all $g \in G$ such that $g \neq 1$ then $\chi_G(g) = 0$.*

Proof. Let B_G be a basis of $\widehat{\mathbb{C}[G]}$ and let ϕ be the corresponding representation. If $\phi(g) = A$ where A is a $|G| \times |G|$ matrix, then the element $a_{ij} = 0$; unless $gg_i = g_{ij}$, in which case

$a_{ij} = 1$. Since $gg_i = g_i$ if and only if $g = 1$, we have $\chi_G(g) = 0$ for $g \neq 1$, it follows that $\chi_G(1) = |G|$. $\square$

Lemma 4.14. [JL01, p. 119] *$\chi_G = \sum_{i=1}^{k} \chi_i(1)\chi_i$ where the χ_i are the irreducible characters of G.*

Proof. As the regular module of G breaks down into k irreducible representations $\phi_1, \ldots, \phi_k$ and each representation ϕ_i occurs $\chi_i(1)$ times, the result follows immediately. $\square$

Example 22. *Trivial and Regular Characters of $Sym(3)$*
Let $G = Sym(3)$ and define $\iota : G \to \mathbb{C}$ by $\iota : g \mapsto 1$; then the trivial character is given by $\chi_0(g) = \mathrm{tr}(1) = 1$ for every element in G.
Recall Example 14 where we calculated the regular representation of $Sym(3)$, we shall list the identity and the generators (which also happen to be representative elements for their respective classes); then calculate the regular character for these elements.

$$\phi(1) = I_6,\ \phi((12)) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \phi((123)) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have $\chi_G(1) = \mathrm{tr}(\phi(1)) = \mathrm{tr}(I_6) = 6 = |G|$, $\chi_G((12)) = \mathrm{tr}(\phi((12)) = 0$ and $\chi_G((123)) = \mathrm{tr}(\phi((123)) = 0$, verifying Lemma 4.13 in this case.

Given any character χ of G we can obtain a new character $\bar{\chi}$ by taking the complex conjugate of χ; of course if $\chi : G \to \mathbb{R}$ then $\chi = \bar{\chi}$. We call this character the *conjugate character.*

Example 23. *The Conjugate Character of C_3*
Let $C_3 = \langle g\colon g^3 = 1\rangle$ and define a representation $\phi : C_3 \to \mathbb{C}$ by $\phi : g^k \mapsto \omega^k$ where $\omega = e^{\frac{2i\pi}{3}}$. Now, let χ_ϕ be the character afforded by ϕ, we see that $\chi_\phi(g^k) = \omega^k$. The conjugate character of χ_ϕ is given by $\overline{\chi_\phi}(g^k) = \overline{\omega^k} = \omega^{3-k}$.

Theorem 4.15. [JL01, p. 123] *Let G be a finite group, let $g \in G$ and let g have order n. Let χ be the character afforded by a $\mathbb{C}[G]$-module M then $\chi(g)$ is the sum of nth roots of unity and $\chi(g^{-1}) = \overline{\chi}(g)$.*

Proof. By Theorem 4.12 there is a basis B of M such that $[g]_B$ is diagonal and each element on the diagonal is an nth root of unity. Therefore, $\chi(g)$ is a sum of nth roots of unity. Consider $[g^{-1}]_B$. Here every root of unity ω_i on the diagonal will be replaced by ω^{-1}. Hence, $\chi(g^{-1}) = \sum_{i=1}^{n} \omega_i^{-1}$. As every nth root of unity satisfies $\omega^{-1} = \overline{\omega}$ the result follows. $\square$

Corollary 4.16. [JL01, p. 123] *If g is conjugate to g^{-1} then $\chi(g) \in \mathbb{R}$.*

Proof. If g is conjugate to g^{-1} then $\chi(g) = \chi(g^{-1}) = \overline{\chi}(g)$. Hence, $\chi(g)$ is a real number. $\square$

4.3.2 Permutation Characters

We explored the permutation representation in Example 15, we will now study the associated character. Let G be a permutation group that acts on a set $\Delta = \{1, \ldots, n\}$ and let $B_\Delta = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis; then $g \in G$ acts on B_Δ by applying the action to the index of the basis vector g is acting on. We extend this action linearly to the whole vector space. Denote the corresponding representation $\hat{P}$.

Let the matrix corresponding to g with respect to B_Δ be denoted $[g]_{B_\Delta}$. As $[g]_{B_\Delta}$ is a permutation matrix then $\operatorname{tr}([g]_{B_\Delta}) = \operatorname{Fix}(g)$ where $\operatorname{Fix}(g) = \{i \in B_\Delta \colon gi = i\}$; that is $\operatorname{Fix}(g)$ is the elements of B_Δ that are fixed by g. This representation has a character χ_Δ and $\chi_\Delta(g) = \operatorname{Fix}(g)$.

Under any permutation representation the subspace $\langle \mathbf{1} \rangle$, where $\mathbf{1}$ is the column vector with 1 in every entry, is invariant. By Maschke's Theorem (Theorem 3.17) we can then see that $\hat{P} = \langle \mathbf{1} \rangle \oplus P$ for some $\mathbb{C}[G]$-module P. The subspace P has a character χ_P and $\chi_P(g) = \operatorname{Fix}(g) - 1$. Finally, if $G = Sym(n)$ we call this character the *standard character*. Later on we will see that if G is 2-transitive then χ_P is irreducible.

Example 24. *The Standard Character of $Sym(3)$*
Recall in Example 15 we calculated the permutation representation of $Sym(3)$. We will state the representation again here for convenience; define $\phi : Sym(3) \to GL_2(\mathbb{C})$ by:

$$\phi(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \phi((12)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \phi((123)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let χ_P be the standard character; then we can see that the identity fixes all three points so $\chi_P(1) = 3 - 1 = 2$ and $\chi_P(1) = \operatorname{tr}(\phi(1)) - 1 = 2$. Similarly $\chi_P((12)) = 1 - 1 = 0$ and $\chi_P((12)) = \operatorname{tr}(\phi((12))) - 1 = 0$, finally $\chi_P((123)) = 0 - 1 = -1$ and $\chi_P((123)) = \operatorname{tr}(\phi((123))) - 1 = -1$.

For any group with both even and odd permutations we can define a character which corresponds to the Sgn representation of the group. We call this character the *Sgn character* and denote the character χ_{Sgn}. If g is an even permutation, then $\chi_{Sgn}(g) = 1$; if g is an odd permutation, then $\chi_{Sgn}(g) = -1$.

4.3.3 Lifted Characters

Let G be a group with a normal subgroup N and let $\nu : G \to G/N$ be the quotient map. Let χ be a character of G/N then the composite map $\chi(\nu(g))$ is a character of G, we call this character the *lift* of χ. The Sgn character of the symmetric group is a specific example of this, as it is the character lifted from $Sym(n)/Alt(n) \cong C_2$. In general, if G has an index 2 subgroup, we can lift the Sgn character.

Definition 4.17. [JL01, p. 124] *Kernel of a Character*
Let χ be a character of a group G, then $\mathrm{Ker}(\chi) = \{g \in G \colon \chi(g) = \chi(1)\}$.

Lemma 4.18. [JL01, p. 169] *Irreducible characters of G/N correspond to irreducible characters of G whose kernel contains N.*

Proof. Let $\nu : G \to G/N$ be the canonical homomorphism and let χ be a character of G/N. The composite map $\chi\nu$ shows that all characters of G/N are characters of G. Moreover, $\chi(\nu(N)) = \chi(1)$ and for $n \in N$ we have:

$$\chi(\nu(n)) = \chi(nN) = \chi(N) = \chi(1).$$

Therefore, $N \leq \mathrm{Ker}(\chi)$. Now we need to check that irreducible characters remain irreducible once they have been *lifted* to G. Let ϕ be the representation that affords the lift of χ and let ψ be the representation that affords χ; let M be a subspace of $\mathbb{C}^n$ and let $\mathbf{m} \in M$. Then $\phi(g)\mathbf{m} \in M$ if and only if $\psi(gN)\mathbf{m} \in M$. Hence, M is a $\mathbb{C}[G]$-submodule if and only if M is a $\mathbb{C}\left[G/N\right]$-submodule. It follows that ψ is irreducible if and only if ϕ is irreducible. □

Lemma 4.19. [JL01, p. 124] *Let χ be a character afforded by ϕ, a representation of a group G and let $g \in G$. Then $|\chi(g)| = \chi(1)$ if and only if $\phi(g) = cI_k$ where $c \in \mathbb{C}$.*

Proof. Suppose that $|\chi(g)| = \chi(1)$, then by Theorem 4.12 we have basis B for $\mathbb{C}^k$ such that $[g]_B$ is a diagonal matrix and the non zero entries ω_i are nth roots of unity. We see that:

$$|\chi(g)| = \left|\sum_{i=1}^{k} \omega_i\right| = \chi(1) = n.$$

Since $|\omega_i| = 1$ for every i it follows that $\omega_i = \omega_j$ for every i and j. Thus, $[g]_B = \omega_1 I_k$. Conversely, let $\phi(g) = cI_k$ where $c \in \mathbb{C}$. Then c is an nth root of unity and $\chi(g) = kc$ so $|\chi(g)| = |kc| = k = \chi(1)$. □

Lemma 4.20. [JL01, p. 124] *Let ϕ be a representation of G with character χ then* $\mathrm{Ker}(\phi) = \mathrm{Ker}(\chi)$.

Proof. If $g \in \text{Ker}(\phi)$ then $\phi(g) = I_n$ and $\chi(g) = \text{tr}(I_n) = \chi(1)$, hence, $\text{Ker}(\phi) \leq \text{Ker}(\chi)$. To show the reverse inclusion suppose that $\chi(g) = \chi(1)$, then $\phi(g) = cI_n$ for some $c \in$. Hence, $\chi(g) = c\chi(1)$, therefore $c = 1$ and $\phi(g) = I_n$ and $g \in \text{Ker}(\phi)$. □

Theorem 4.21. [JL01, p. 124] *Let χ be a character afforded by a representation ϕ of a group G then* $\text{Ker}(\chi) \trianglelefteq G$.

Proof. By Lemma 4.20 we have $\text{Ker}(\chi) = \text{Ker}(\phi)$; as representations of a group are exactly group homomorphisms into $\text{GL}(V)$ for some vector space V, it follows that $\text{Ker}(\chi) \trianglelefteq G$. □

Theorem 4.22. [JL01, p. 172] *A group G is simple if and only if $\chi(g) \neq \chi(1)$ for every non-trivial irreducible character χ and every non identity element g.*

Proof. Suppose for some non trivial irreducible character χ and for some $g \in G$ such that $g \neq 1$ we have $\chi(g) = \chi(1)$; then $g \in \text{Ker}(\chi)$. Let ϕ be the representation affording χ, then, by Lemma 4.20, we have $\text{Ker}(\chi) = \text{Ker}(\phi)$. As ϕ is irreducible $\text{Ker}(\phi) \neq G$, it follows that $\text{Ker}(\chi) \neq G$. Hence, $\text{Ker}(\chi)$ is a non trivial normal subgroup of G and G is not simple.

Conversely, suppose for contradiction G is not simple but $\text{Ker}(\chi) = \{1\}$ for every irreducible character χ of G. As G is not simple, there exists a non trivial normal subgroup N of G. As $N \trianglelefteq G$ there is a non trivial irreducible character χ of G lifted from G/N. But then by Lemma 4.18, $N \leq \text{Ker}(\chi)$, a contradiction. Therefore, N must be trivial and G is simple. □

4.4 The Character Table

The *Character Table* of a group G is a table whose columns correspond to the conjugacy classes of the group and whose rows correspond to the irreducible representations. The entries of the table are the characters evaluated for each conjugacy class.

Example 25. *The Character Tables of Cyclic Groups.*
We shall begin by calculating a representation of any cyclic group of order n. Let $C_n = \langle g\colon g^n = 1 \rangle$ and define $\phi_n : C_n \to \mathbb{C}$ by $\phi_n : g^k \mapsto e^{\frac{2i\pi k}{n}}$. Let χ_{ϕ_n} be the character afforded by ϕ_n then $\chi_{\phi_n}(g) = \phi_n(g)$. We shall calculate the table for C_2 first.

Now, C_2 is of order 2 and has two conjugacy classes, hence we are looking for two, 1-dimensional irreducible representations. The representations are obviously ϕ_2 and the trivial representation ι. We can now write down the character table of C_2 (Table 4.1).

C_2	1	g
χ_0	1	1
χ_{ϕ_2}	1	-1

Table 4.1: The character table of C_2.

Next, we will calculate the character table of C_3, here we have 3 conjugacy classes, so we are looking for three, one dimensional irreducible representations. Here, our first two representations are ϕ_3 and the trivial representation. However, $\chi_{\phi_3}(g)$ takes a complex value, we can get the third irreducible character by taking complex conjugates of χ_{ϕ_3}. Let $\omega = e^{\frac{2i\pi}{3}}$, then we can write down the complete character table (Table 4.2).

C_3	1	g	g^2
χ_0	1	1	1
χ_{ϕ_3}	1	ω	ω^2
$\overline{\chi_{\phi_3}}$	1	ω^2	ω

Table 4.2: The character table of C_3, where $\omega = e^{\frac{2i\pi}{3}}$.

We can construct every non trivial irreducible character of a cyclic group C_n by defining group homomorphisms $\psi_a \colon g^k \mapsto e^{\frac{2i\pi(k+a)}{n}}$ where $a \in \mathbb{Z}$ and $0 \leq a \leq n$.

Example 26. *The Character Table of* $Sym(3)$
Recall that we have found three irreducible representations of $Sym(3)$: the trivial representation, the Sgn representation and a 2-dimensional representation. A quick calculation will show that $Sym(3)$ has three conjugacy classes $\{1\}$, $\{(12),(13),(23)\}$ and $\{(123),(132)\}$ and so we know our three irreducible representations are all of them.

Let 1, (12) and (123) be representative elements for their respective conjugacy classes. We can now calculate our character table. Firstly consider the trivial representation ι which yields the trivial character χ_0, we know that $\chi_0(g) = 1$ for every $g \in Sym(3)$. Next, we will calculate the Sgn character χ_{Sgn}:

$$\chi_{Sgn}(1) = 1,\ \chi_{Sgn}((12)) = -1 \text{ and } \chi_{Sgn}((123)) = 1.$$

Finally, we need to calculate the 2-dimensional character χ_3, this is actually the *standard character* of $Sym(3)$ so we just need to calculate $fix(g) - 1$ for each conjugacy class,

$$\chi_3(1) = 2,\ \chi_3((12)) = 0 \text{ and } \chi_3((123)) = -1.$$

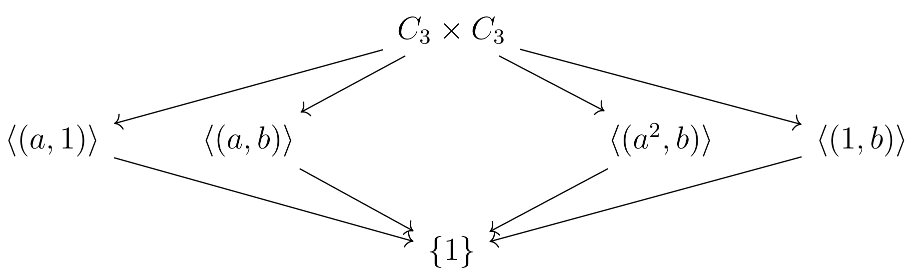
We can now write down the complete character table (Table 4.3).

$Sym(3)$	1	(12)	(123)
χ_0	1	1	1
χ_{Sgn}	1	-1	1
χ_2	2	0	-1

Table 4.3: The character table of $Sym(3)$.

Example 27. *The Character Table of* $C_3 \times C_3$

In this example we will calculate the character table of the direct product of two cyclic groups of order 3. First, recall that $C_3 \times C_3 = \{(a,b) \colon a, b \in C_3\}$, in fact this is exactly a two dimensional vector space over $\mathbb{F}_3$. Before we calculate the characters we will look at the subgroups; we can find four copies of C_3 within $C_3 \times C_3$, there are the two obvious subgroups generated by $(a, 1)$ and $(1, b)$, however, there are two more, generated by (a, b) and (a^2, b). We shall summarize this with a diagram of containments:



We can use these subgroups along with our knowledge of C_3 to calculate all of the characters of $C_3 \times C_3$. As our group is abelian of order 9 it has 9 conjugacy classes and 9 irreducible representations. We can write every representation down explicitly, let $\omega = e^{\frac{2i\pi}{3}}$ and ι be the trivial representation of $C_3 \times C_3$. Using the aforementioned subgroup calculations we can now define functions to lift characters from each of the subgroups. Define $\psi_i : C_3 \times C_3 \to \mathbb{C}$ by:

$$\psi_1((a^k, b^l)) \mapsto \omega^k,\ \psi_2((a^k, b^l)) \mapsto \omega^l,\ \psi_3((a^k, b^l)) \mapsto \omega^{k+l} \text{ and } \psi_4((a^k, b^l)) \mapsto \omega^{k-l}.$$

The remaining four one dimensional representations are given by taking the complex conjugates of the ψ_i. Let χ_i be the character afforded by ψ_i; we can now write down the completed character table (Table 4.4).

$C_3 \times C_3$	1	$(a,1)$	$(a^2,1)$	$(1,b)$	$(1,b^2)$	(a,b)	(a^2,b^2)	(a^2,b)	(a,b^2)
χ_0	1	1	1	1	1	1	1	1	1
χ_1	1	ω	ω^2	1	1	ω	ω^2	ω^2	ω
$\overline{\chi_1}$	1	ω^2	ω	1	1	ω^2	ω	ω	ω^2
χ_2	1	1	1	ω	ω^2	ω	ω^2	ω	ω^2
$\overline{\chi_2}$	1	1	1	ω^2	ω	ω^2	ω	ω^2	ω
χ_3	1	ω	ω^2	ω	ω^2	ω^2	ω	1	1
$\overline{\chi_3}$	1	ω^2	ω	ω^2	ω	ω	ω^2	1	1
χ_4	1	ω	ω^2	ω	ω^2	1	1	ω	ω^2
$\overline{\chi_4}$	1	ω^2	ω	ω^2	ω	1	1	ω^2	ω

Table 4.4: The character table of $C_3 \times C_3$, where $\omega = e^{\frac{2i\pi}{3}}$.

4.5 Schur Orthogonality Relations

Definition 4.23. [JL01, p. 143] *Constituent Character*
Let $\chi_1, \dots, \chi_k$ be the irreducible characters of a group G. Let χ be any character of G then $\chi = \sum_{i=1}^{k} c_i \chi_i$. Wherever $c_i \neq 0$ we say that χ_i is a *constituent* of χ.

Definition 4.24. [JL01, p. 134] *(Complex) Inner Product Space*
Let V be a complex vector space and let $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{C}$ be a function that satisfies the following for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\forall c \in \mathbb{C}$:

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$;
2. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$;
3. $\langle \mathbf{u} + \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle$;
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

Lemma 4.25. [JL01, p. 134] *Let V be the space of functions from a group G to $\mathbb{C}$ and let $\phi, \psi \in V$. Define $\langle \cdot, \cdot \rangle$ by:*

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi}(g). \tag{4.2}$$

Then V with $\langle \cdot, \cdot \rangle$ is an inner product space.

Proof. This is immediate from the definition of an inner product space. □

Lemma 4.26. [JL01, p. 135] *Let ϕ and ψ be characters of a finite group G then $\langle \phi, \psi \rangle = \langle \psi, \phi \rangle$.*

Proof. Since $\phi(g^{-1}) = \overline{\phi}(g)$ and $\{g^{-1} : g \in G\} = G$ we obtain

$$\langle \phi, \psi \rangle = \sum_{g \in G} \phi(g)\overline{\psi}(g) = \sum_{g \in G} \phi(g)\psi(g^{-1}) = \sum_{g \in G} \phi(g^{-1})\psi(g) = \sum_{g \in G} \overline{\phi(g)}\psi(g) = \langle \psi, \phi \rangle.$$

□

Lemma 4.27. [JL01, p. 138] *Let $\mathbb{C}[G] = M_1 \oplus M_2$ (so M_1 and M_2 are sums of non isomorphic submodules). Let χ be the character afforded by M_1 and let $\mathbf{1} = \mathbf{e}_1 + \mathbf{e}_2$, then*

$$\mathbf{e}_1 = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1})g.$$

Proof. Let $x \in G$, then the function $\Phi : \mathbb{C}[G] \to \mathbb{C}[G]$ by $\Phi : \mathbf{a} \mapsto x^{-1}\mathbf{e}_1\mathbf{a}$ is an endomorphism of $\mathbb{C}[G]$. For $\mathbf{m}_1 \in M_1$ and $\mathbf{m}_2 \in M_2$, because all $\mathbb{C}[G]$-homomorphisms from M_1 to M_2 are trivial, we see that $\Phi(\mathbf{m}_1) = x^{-1}\mathbf{e}_1\mathbf{m}_1 = x^{-1}\mathbf{e}_1$ and $\Phi(\mathbf{m}_2) = x^{-1}\mathbf{e}_2\mathbf{m}_2 = \mathbf{0}$. It follows that $\mathrm{tr}(\Phi) = \chi(x^{-1})$ where χ is the character afforded by M_1.

As $\mathbf{e}_1 \in \mathbb{C}[G]$ we see that $\mathbf{e_1} = \sum_{g \in G} c_g g$, where $c_g \in \mathbb{C}$. Recall that the regular character $\chi_G(g)$ takes the value $|G|$ if $g = 1$ and 0 otherwise. Hence, $\Phi : \mathbf{a} \mapsto x^{-1} \sum_{g \in G} c_g g\mathbf{a}$ and we see that $\mathrm{tr}(\Phi) = c_x|G|$. It follows that $c_x = \chi(x^{-1})/|G|$, hence, $\mathbf{e}_1 = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1})g$. □

Lemma 4.28. [JL01, p. 139] *Let M_1 be the same as in Lemma 4.27 and let χ be the character afforded by M_1 then $\langle \chi, \chi \rangle = \chi(1)$.*

Proof. The coefficient of $1 \in G$ in $\mathbf{e}_1^2$ is $\frac{1}{|G|^2} \sum_{g \in G} \chi(g^{-1})\chi(g) = \frac{1}{|G|}\langle \chi, \chi \rangle$, this can be seen by applying the previous lemma to the definition of multiplication in the group algebra. Now, let $\mathbf{m}_1 \in M_1$ and $\mathbf{m}_2 \in M_2$ and observe that $\mathbf{m}_1 = \mathbf{m}_1 1 = \mathbf{m}_1(\mathbf{e}_1 + \mathbf{e}_2) = \mathbf{m}_1\mathbf{e}_1$, taking $\mathbf{m}_1 = \mathbf{e}_1$ gives $\mathbf{e}_1 = \mathbf{e}_1^2$. Now, the coefficient of $1 \in G$ in $\mathbf{e}_1$ is $\frac{1}{|G|}\chi(1)$; equating these gives $\langle \chi, \chi \rangle = \chi(1)$. □

Theorem 4.29. [JL01, p. 140] *Let χ_i and χ_j be irreducible characters of a group G, afforded by non isomorphic $\mathbb{C}[G]$-modules M_i and M_j; then, $\langle \chi_i, \chi_j \rangle = \delta_{ij}$, where δ_{ij} is Dirac's Delta Function given by* $\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j; \\ 1 \text{ if } i = j. \end{cases}$

Proof. Let $k = \dim(M_i)$ and define M to be the direct sum of the k isomorphic copies of M_i in $\mathbb{C}[G]$. Let every other $\mathbb{C}[G]$-submodule be contained in N, then $\mathbb{C}[G] = M \oplus N$. Note that M and N intersect trivially, it follows that the character afforded by M is $k\chi_i$. By Lemma 4.28 we obtain $\langle k\chi_i, k\chi_i \rangle = k\chi_i(1)$. As $\chi_i(1) = k$ we see that $\langle \chi_i, \chi_i \rangle = 1$.

Let $\dim(M_j) = l$ and let X be the direct sum of the k isomorphic copies of M_i and the l isomorphic copies of M_j. Let every other $\mathbb{C}[G]$-submodule be contained in Y, then

$\mathbb{C}[G] = X \oplus Y$. Here X and Y intersect trivially, so the character of X is $k\chi_i(1) + l\chi_j(1)$. By Lemma 4.28 we see that

$$\begin{aligned} k\chi_i(1) + l\chi_j(1) &= \langle k\chi_i + l\chi_j, k\chi_i + l\chi_j \rangle \\ &= k^2\langle \chi_i, \chi_i \rangle + l^2\langle \chi_j, \chi_j \rangle + kl(\langle \chi_i, \chi_j \rangle + \langle \chi_j, \chi_i \rangle) \\ &= k^2 + l^2 + kl(\langle \chi_i, \chi_j \rangle + \langle \chi_j, \chi_i \rangle). \end{aligned}$$

As $\chi_i(1) = k$ and $\chi_j(1) = l$, we see that $\langle \chi_i, \chi_j \rangle + \langle \chi_j, \chi_i \rangle = 0$ and as $\langle \chi_i, \chi_j \rangle = \langle \chi_j, \chi_i \rangle$ we obtain $\langle \chi_i, \chi_j \rangle = 0$. □

Theorem 4.30. [JL01, p. 143] *Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of a finite group G. Let χ be any character of G then $\chi = \sum_{i=1}^{k} c_i\chi_i$ where $0 \leq c_i \in \mathbb{Z}$. Moreover, $c_i = \langle \chi, \chi_i \rangle$ and $\langle \chi, \chi \rangle = \sum_{i=1}^{k} c_i^2$.*

Proof. The statement $\chi = \sum_{i=1}^{k} c_i\chi_i$ where the c_i are non negative integers, follows immediately from Maschke's Theorem. Now, let there be c_i copies of χ_i in χ; then as $\langle \chi_i, \chi_j \rangle = \delta_{ij}$ it follows that $\langle \chi, \chi_i \rangle = c_i$. The last statement is immediate. □

Theorem 4.31. [JL01, p. 143] *Let G be a finite group and let M be a $\mathbb{C}[G]$-module. Let χ be the character afforded by M; then M is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*

Proof. If M is irreducible then by Theorem 4.29 we have $\langle \chi, \chi \rangle = 1$. Let $\chi = c_1\chi_1 + \cdots + c_k\chi_k$ where c_i is a non negative integer. Now, assume $\langle \chi, \chi \rangle = 1$ then $\langle \chi, \chi \rangle = c_1^2 + \cdots + c_k^2 =$ which implies that one of the $d_i = 1$ and the rest equal 0, and so χ is irreducible. □

Definition 4.32. [JL01, p. 152] *Class Function*
We define a *class function* on G, as a function $\phi : G \to \mathbb{C}$ such that $\mathrm{Im}(\phi)$ is constant for each conjugacy class. Characters are an example of class functions.

Lemma 4.33. [JL01, p. 153] *Every class function of G can be expressed as a linear combination of the irreducible characters of G.*

Proof. Let G have k conjugacy classes and observe that by Theorem 4.10, G has k irreducible characters $\chi_1, \ldots, \chi_k$. The set of class functions of G forms a vector space over $\mathbb{C}$ of dimension k (the number of classes). Now, assume that $c_1\chi_1 + \ldots + c_k\chi_k = 0$ for some complex numbers c_i. For every i we have $0 = \langle c_1\chi_1 + \ldots + c_k\chi_k, \chi_i \rangle = c_i$ and so the irreducible characters are linearly independent. Thus, the set of irreducible characters form a basis for the space of class functions. □

Theorem 4.34. [JL01, p. 154] *Let $g \in G$, then g is conjugate to g^{-1} if and only if $\chi(g) \in \mathbb{R}$ for all characters χ of G.*

Proof. If g is conjugate to g^{-1}, then $\chi(g) = \chi(g^{-1})$ by Corollary 4.16. Conversely, suppose $\chi(g) \in \mathbb{R}$ for all characters χ of G, then we have $\chi(g) = \chi(g^{-1})$. It follows that for every class function ϕ on G, we have $\phi(g) = \phi(g^{-1})$. Now, define a class function ζ that takes the value 1 on the conjugacy class of g and 0 elsewhere. Observe that $\zeta(g) = \zeta(g^{-1})$, thus g is conjugate to g^{-1}. $\square$

Theorem 4.35. [JL01, p. 161] Schur's Orthogonality Relations
Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G and let $g_1, \ldots, g_k$ be representative elements of the conjugacy classes of G, then for every $r, s \in \{1, \ldots, k\}$ we have:

$$\sum_{i=1}^{k} \frac{\chi_r(g_i)\overline{\chi_s}(g_i)}{|C_G(g_i)|} = \delta_{rs}; \tag{4.3}$$

$$\sum_{i=1}^{k} \chi_i(g_r)\overline{\chi_i}(g_s) = \delta_{rs}|C_G(g_r)|. \tag{4.4}$$

Proof. Let G have k conjugacy classes and let g_i^G be the conjugacy class of G containing g_i then

$$\sum_{g \in G} \chi_s(g)\overline{\chi_r}(g) = \sum_{i=1}^{k} |g_i^G|\chi_s(g_i)\overline{\chi_r}(g_i).$$

As G is the union of its k conjugacy classes then by Theorem 2.9 we have $|g_i^G| = |G|/|C_G(g_i)|$. Now, observe that

$$\begin{aligned}
\langle \chi_s, \chi_r \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_r(g)\overline{\chi_s}(g) \\
&= \frac{1}{|G|} \sum_{i=1}^{k} \sum_{g \in g_i^G} \chi_r(g)\overline{\chi_s}(g) \\
&= \sum_{i=1}^{k} \frac{|g_i^G|}{|G|} \chi_r(g)\overline{\chi_s}(g) \\
&= \sum_{i=1}^{k} \frac{\chi_r(g)\overline{\chi_s}(g)}{|C_G(g_i)|}.
\end{aligned}$$

Applying Theorem 4.29 we arrive at (4.3).

Now, let ϕ_s be a class function of G that satisfies $\phi_s(g_r) = \delta_{rs}$, say $\phi_s = \sum_{i=1}^{k} c_i\chi_i$ then $c_i = \langle \phi_s, \chi_i \rangle$. Now, $\phi_s(g) = 1$ when g is conjugate to g_s and $\phi_s(g) = 0$ otherwise. There are

$|g_s^G| = |G|/|C_G(g_s)|$ elements conjugate to g_s, hence,

$$c_i = \frac{1}{|G|}\sum_{g\in g_s^G}\phi_s(g)\overline{\chi_i}(g) = \frac{\overline{\chi_i}(g_s)}{|C_G(g_s)|}.$$

Finally,

$$\delta_{rs} = \phi_s(g_r) = \sum_{i=1}^{k} c_i\chi_i(g_r) = \sum_{i=1}^{k}\frac{\chi_i(g_r)\overline{\chi_i}(g_s)}{|C_G(g_s)|},$$

multiplying both sides by $|C_G(g_s)|$ gives (4.4). □

Lemma 4.36. [JL01, p. 340] *Let G be a group acting on a finite set Ω, let χ be the character afforded by the permutation representation and let χ_0 be the trivial character. Let $\mathrm{Orb}(\Omega)$ be the set of orbits of Ω then we have:*

$$\langle\chi,\chi_0\rangle = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)| = |\mathrm{Orb}(\Omega)|.$$

Proof. Let $\Lambda = \{(g,\omega)\colon g\in G,\ \omega\in\Omega\ \ g\omega=\omega\}$. It is immediate that $|\Lambda| = \sum_{g\in G}|\mathrm{Fix}(g)|$. By considering the stabilisers of points in Ω we also see that $|\Lambda| = \sum_{\omega\in\Omega}|G_\omega|$. Now, recall that for $\omega\in\Omega$ we have $|G| = |\omega^G||G_\omega|$. Let G have k orbits on Ω then

$$|\Lambda| = \sum_{\omega\in\Omega}|G_\omega| = \sum_{i=1}^{k}|\omega_i^G||G_{\omega_i}| = k|G| = |\mathrm{Orb}(\Omega)||G|.$$

Observe that $\langle\chi,\chi_0\rangle = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|$, the result follows. □

We remark that a finite group G acts 2-transitively on a set Ω if $|\mathrm{Orb}(\Omega\times\Omega)| = 2$. These two orbits are exactly, the set of unique pairs $\Omega^{(2)}$ and the set $\{(\omega,\omega)\colon \omega\in\Omega\}$.

Theorem 4.37. [JL01, p. 342] *If G acts 2-transitively on a finite set Ω, then the character given by $\chi(g) = |\mathrm{Fix}(g)| - 1$ is irreducible.*

Proof. Let the action of G on Ω have character χ_Ω. Consider the action of G on $\Omega\times\Omega$. Observe that,

$$\langle\chi_\Omega,\chi_\Omega\rangle = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)||\mathrm{Fix}(g)| = |\mathrm{Orb}(\Omega\times\Omega)|.$$

As G is 2-transitive we have $\langle\chi_\Omega,\chi_\Omega\rangle = |\mathrm{Orb}(\Omega\times\Omega)| = 2$. Finally, by Lemma 4.36, $\langle\chi_\Omega,\chi_0\rangle = 1$. The result follows. □

In Example 12 we explored a representation of Q_8 over $\mathbb{F}_3$. In the same example we also investigated the group He(3). We will now calculate the complex character tables of both groups.

Example 28. *The Character Table of the Quarternion Group Q_8*
Recall that the Quarternion group Q_8 has the following presentation,

$$\langle -1, i, j, k\colon -1^2 = 1, i^2 = j^2 = k^2 = ijk = -1\rangle.$$

Here each of the i, j and k generate a subgroup of order 4. As $|Q_8| = 8$ each of the order 4 subgroups are index 2 and therefore normal; the quotient groups formed are all isomorphic to C_2. We can lift a character from each C_2, in each case the kernel will be exactly $\langle i\rangle$, $\langle j\rangle$ or $\langle k\rangle$.

Recall that Q_8 has five conjugacy classes, we have just calculated three linear characters. A fourth character is simply the trivial character, that means that the last character must have dimension 2 as $2^2 = 8 - 4$. Consider the character table so far.

Q_8	1	-1	i	j	k
χ_0	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_4	2				

We can now use Equation 4.4 to calculate the remaining row. Let $g_1 = 1$ and $g_2 = -1$, as $g_1 \neq -1$ the right hand side of Equation 4.4 is equal to 0. Let $\chi_4(-1) = x$ then,

$$\begin{aligned} \sum_{i=0}^{4} \chi(1)\chi(-1) &= 0 \\ 1\times 1 + 1\times 1 + 1\times 1 + 1\times 1 + 2x &= 0 \\ x &= -2 \end{aligned}$$

Similar arguments give $\chi_4(i) = \chi_4(j) = \chi_4(k) = 0$. Recall the conjugacy class labels from Table 2.2. The full character table for Q_8 is given in Table 4.5.

Q_8	$C1$	$C2$	$C4_A$	$C4_B$	$C4_C$
χ_0	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_4	2	-2	0	0	0

Table 4.5: The character table of Q_8.

Example 29. *The Character Table of* He(3)
The *Heisenberg group* over a field $\mathbb{F}$ is isomorphic to the unitriangular matrices over $\mathbb{F}$. That is,

$$\mathrm{He}(3) = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{F}_3 \right\}.$$

Let $G = \mathrm{He}(3)$, it is easy to see that $|G| = 27$ as there are three choices for each of x, y and z. Moreover, elements where both x and y equal 0 are in the centre of the group $\mathrm{Z}(G)$; there are 3 such elements, each in their own conjugacy class and we will index them by z.

The remaining conjugacy classes can be found by direct calculation and the elements are listed in Table 4.6. We will index each of these classes by (x, y) and note that the centraliser in each case has order 9. This gives us the 11 conjugacy classes of He(3).

Class	Centraliser	Elements
1	27	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
2	27	$\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$
3	27	$\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}$
$(0,1)$	9	$\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&0&2\\0&1&1\\0&0&1\end{bmatrix}$
$(0,2)$	9	$\begin{bmatrix}1&0&0\\0&1&2\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&0&1\\0&1&2\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&0&2\\0&1&2\\0&0&1\end{bmatrix}$
$(1,0)$	9	$\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&1&2\\0&1&0\\0&0&1\end{bmatrix}$
$(1,1)$	9	$\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&1&2\\0&1&1\\0&0&1\end{bmatrix}$
$(1,2)$	9	$\begin{bmatrix}1&1&0\\0&1&2\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&1&1\\0&1&2\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&1&2\\0&1&2\\0&0&1\end{bmatrix}$
$(2,0)$	9	$\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&2&1\\0&1&0\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&2&2\\0&1&0\\0&0&1\end{bmatrix}$
$(2,1)$	9	$\begin{bmatrix}1&2&0\\0&1&1\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&2&1\\0&1&1\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&2&2\\0&1&1\\0&0&1\end{bmatrix}$
$(2,2)$	9	$\begin{bmatrix}1&2&0\\0&1&2\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&2&1\\0&1&2\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&2&2\\0&1&2\\0&0&1\end{bmatrix}$

Table 4.6: The conjugacy classes of He(3).

As $\mathrm{Z}(G) \trianglelefteq G$ we can lift characters from $G/Z(G) \cong C_3^2$. To prove this is an isomorphism, first note that each coset is indexed by (x, y). Now define:

$$\nu\colon G \to C_2^3 \text{ by } \nu\colon \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto (a^x, b^y).$$

Observe that for every $g_1, g_2 \in G$ we have:

$$\nu(g_1 g_2) = \nu\left(\begin{bmatrix} 1 & x_1 & z_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & x_2 & z_2 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{bmatrix}\right) = \nu\left(\begin{bmatrix} 1 & x_1 + x_2 & z_1 + x_1 y_2 + z_2 \\ 0 & 1 & y_{+} y_2 \\ 0 & 0 & 1 \end{bmatrix}\right)$$
$$= (a^{x_1+x_2}, b^{y_1+y_2}) = (a^{x_1}, b^{y_1})(a^{x_2}, b^{y_2}) = \nu(g_1)\nu(g_2).$$

Hence, ν is a group homomorphism.

We can now lift the 9 linear characters from C_3^2. As $Ker(\nu) = Z(G)$, the central elements will take character values of 1 for the lifted characters. We know that $|G| = 27$, so we can deduce that the two other characters have dimension 3, we will call these χ_5 and χ_6.

We can use (4.4) on column (x, y) to calculate χ_5 and χ_6 for the classes indexed by (x, y). Let $\alpha_{x,y} = \chi_5((x, y))$ and $\beta_{x,y} = \chi_6((x, y))$, then we see that $\alpha_{x,y}\overline{\alpha_{x,y}} + \beta_{x,y}\overline{\beta_{x,y}} = 0$ which implies that $|\alpha_{x,y}|^2 + |\beta_{x,y}|^2 = 0$, hence, $\alpha_{x,y} = \beta_{x,y} = 0$.

He(3)	0	1	2	$(1,0)$	$(2,0)$	$(0,1)$	$(0,2)$	$(1,1)$	$(2,2)$	$(2,1)$	$(1,2)$
χ_0	1	1	1	1	1	1	1	1	1	1	1
χ_1	1	1	1	ω	ω^2	1	1	ω	ω^2	ω^2	ω
$\overline{\chi_1}$	1	1	1	ω^2	ω	1	1	ω^2	ω	ω	ω^2
χ_2	1	1	1	1	1	ω	ω^2	ω	ω^2	ω	ω^2
$\overline{\chi_2}$	1	1	1	1	1	ω^2	ω	ω^2	ω	ω^2	ω
χ_3	1	1	1	ω	ω^2	ω	ω^2	ω^2	ω	1	1
$\overline{\chi_3}$	1	1	1	ω^2	ω	ω^2	ω	ω	ω^2	1	1
χ_4	1	1	1	ω	ω^2	ω	ω^2	1	1	ω	ω^2
$\overline{\chi_4}$	1	1	1	ω^2	ω	ω^2	ω	1	1	ω^2	ω
χ_5	3	r_1	r_2	0	0	0	0	0	0	0	0
χ_6	3	s_1	s_2	0	0	0	0	0	0	0	0

Let $\chi_5(1) = r_1, \chi_5(2) = r_2, \chi_6(1) = s_1$ and $\chi_6(2) = s_2$. Now, the classes indexed by 1 and 2 are inverse pairs, that is for $g \in 1$ we have $g^{-1} \in 2$. By Theorem 4.15 we see that $r_2 = \overline{r_1}$ and $\chi_6 = \overline{\chi_5}$. Hence, $s_i = \overline{r_i}$.

Using Equation 4.4 on columns 0 and 1 we obtain the equation $r_1 + \overline{r_1} = -3$ Using Equation 4.3 on row χ_5 and $\chi_6 = \overline{\chi_5}$ we obtain the following equation.

$$0 = \frac{9}{27} + \frac{r_1^2}{27} + \frac{\overline{r_1}}{27} + \frac{0^2}{9} + \cdots + \frac{0^2}{9}$$
$$\Rightarrow -9 = r_1^2 + \overline{r_1}^2.$$

Solving the equations for r_1, we obtain $r_1 = \frac{-3\pm3\sqrt{3}i}{2}$. We are free to choose which root to set as r_1, so let r_1 be the positive root; then $\overline{r_1}$ is the negative root. Table 4.7 shows the completed character table of He(3).

He(3)	0	1	2	$(1,0)$	$(2,0)$	$(0,1)$	$(0,2)$	$(1,1)$	$(2,2)$	$(2,1)$	$(1,2)$
χ_0	1	1	1	1	1	1	1	1	1	1	1
χ_1	1	1	1	ω	ω^2	1	1	ω	ω^2	ω^2	ω
$\overline{\chi_1}$	1	1	1	ω^2	ω	1	1	ω^2	ω	ω	ω^2
χ_2	1	1	1	1	1	ω	ω^2	ω	ω^2	ω	ω^2
$\overline{\chi_2}$	1	1	1	1	1	ω^2	ω	ω^2	ω	ω^2	ω
χ_3	1	1	1	ω	ω^2	ω	ω^2	ω^2	ω	1	1
$\overline{\chi_3}$	1	1	1	ω^2	ω	ω^2	ω	ω	ω^2	1	1
χ_4	1	1	1	ω	ω^2	ω	ω^2	1	1	ω	ω^2
$\overline{\chi_4}$	1	1	1	ω^2	ω	ω^2	ω	1	1	ω^2	ω
χ_5	3	α	$\overline{\alpha}$	0	0	0	0	0	0	0	0
χ_6	3	$\overline{\alpha}$	α	0	0	0	0	0	0	0	0

Table 4.7: The character table of He(3), where $\omega = e^{\frac{2i\pi}{3}}$ and $\alpha = \frac{1}{2}(-3 + 3\sqrt{-3})$.

Chapter 5

Characters, Subgroups and Tensor Products

In the previous chapter, we described several methods of obtaining characters of a finite group. For example, lifting characters showed the relationship between a group G and its quotient groups. We will now develop relationships between a group G and its overgroups and subgroups. We will also look at a method of creating new characters from old via tensor products; this will give us all of the tools needed to construct the character tables of the small Mathieu groups.

5.1 Tensor Products of characters

Definition 5.1. *Module Tensor Product*
Let G be a finite group and let M and N be $\mathbb{C}[G]$-modules with basis $\{\mathbf{m}_1, \ldots, \mathbf{m}_k\}$ and $\{\mathbf{n}_1, \ldots, \mathbf{n}_l\}$ respectively. We define the *tensor product of* M *and* N to be $M \otimes N$, the kl dimensional $\mathbb{C}[G]$-module with basis given by the set of symbols $\{\mathbf{m}_i \otimes \mathbf{n}_j \colon 1 \leq i \leq m,\ 1 \leq j \leq n\}$. We define that for every $g \in G$:

$$g(\mathbf{m}_i \otimes \mathbf{n}_j) = g\mathbf{m}_i \otimes g\mathbf{n}_j.$$

Let $f_i, c_i \in \mathbb{C}$, $\mathbf{m} \in M$, $\mathbf{n} \in N$ with $\mathbf{m} = \sum_{i=1}^{k} f_i \mathbf{m}_i$ and $\mathbf{n} = \sum_{j=1}^{l} c_j \mathbf{n}_j$ then

$$\mathbf{m} \otimes \mathbf{n} = \sum_{i,j} f_i c_j (\mathbf{m}_i \otimes \mathbf{n}_j).$$

Theorem 5.2. [JL01, p. 192] *Let G be a finite group and let M and N be $\mathbb{C}[G]$-modules affording characters χ_ϕ and χ_ψ, denote the character of $M \otimes N$ by $\chi_{\phi\psi}$. Then $\chi_{\phi\psi}(g) =$*

$\chi_\phi(g)\chi_\psi(g)$.

Proof. Let $g \in G$. By Theorem 4.12 we can pick a basis $\{\mathbf{m}_1, \dots, \mathbf{m}_k\}$ and $\{\mathbf{n}_1, \dots, \mathbf{n}_l\}$ for M and N respectively such that $g\mathbf{m}_i = c_i\mathbf{m}_i$ and $g\mathbf{n}_j = k_j\mathbf{n}_j$ where $c_i, k_j \in \mathbb{C}$. It follows that $\chi_\phi(g) = \sum_{i=1}^{k} c_i$ and $\chi_\psi(g) = \sum_{j=1}^{l} f_j$. Observe that $g(\mathbf{m}_i \otimes \mathbf{n}_j) = c_i f_j(\mathbf{m}_i \otimes \mathbf{n}_j)$. As the $\mathbf{m}_i \otimes \mathbf{n}_j$ form a basis for $M \otimes N$, it follows that the character $\chi_{\phi\psi}(g) = \sum_{i,j} c_i f_j = \chi_\phi(g)\chi_\psi(g)$. □

We make the remark that for a character χ of a finite group G we define $\chi\chi = \chi^2$ and χ^0 to be the trivial character. The next Theorem will require the use of a Vandermonde matrix. A matrix X is termed a *Vandermonde matrix* when $X_{ij} = x_i^{j-1}$ and each x_i is a distinct complex number. Matrices of this form are invertible [HJ94].

Theorem 5.3. [JL01, p. 195] *Let χ be a character afforded by a faithful representation of a group G. Assume $\chi(g)$ takes r distinct values over every element $g \in G$; then every irreducible character of G is a constituent of one of the powers $\chi^0, \dots, \chi^{r-1}$.*

Proof. Let $x_1, \dots, x_r$ be the r values taken by χ and define $X_i = \{g \in G : \chi(g) = x_i\}$. Let $x_1 = \chi(1)$ then as χ is faithful, $X_1 = \mathrm{Ker}(\chi) = \{1\}$. Let ϕ be an irreducible character of G and let $y_i = \sum_{g \in X_i} \overline{\phi}(g)$; then for every j in $1 \le j \le r-1$ we have

$$\langle \chi^j, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} (\chi(g))^j \, \overline{\phi}(g) = \frac{1}{.}|G| \sum_{i=1}^{r} (x_i^j) y_i \tag{5.1}$$

Let X be the $r \times r$ matrix with entries $X_{ij} = x_i^{j-1}$; then X is a Vandermonde matrix and X^{-1} exists. Now, assume for contradiction that ϕ is not a constituent of any of the χ^j, then (5.1) must equal 0. Now, as X is invertible every this means that every $y_i = 0$, but $y_1 = \phi(1) \neq 0$ and the result follows. □

The previous theorem gives us a method of potentially finding the irreducible characters of group G given a faithful character χ. The problem is that we do not have a method of breaking apart χ^2. The following theorems will develop such a method.

Definition 5.4. [JL01, p. 196] *Symmetric and Antisymmetric Modules*
Let G be a finite group and M be a $\mathbb{C}[G]$-module and define $T : M \otimes M \to M \otimes M$ by $T : \mathbf{m}_i \otimes \mathbf{m}_j = \mathbf{m}_j \otimes \mathbf{m}_i$ extended linearly. We define the *symmetric submodule* of $M \otimes M$ as:

$$S(M \otimes M) = \{\mathbf{m} \in M \otimes M : T(\mathbf{m}) = \mathbf{m}\}.$$

We define the *antisymmetric submodule* of $M \otimes M$ as:

$$A(M \otimes M) = \{\mathbf{m} \in M \otimes M : T(\mathbf{m}) = -\mathbf{m}\}.$$

Theorem 5.5. [JL01, p. 196] *Let G be a finite group and M be a $\mathbb{C}[G]$-module, then the module $M \otimes M$ decomposes into $S(M \otimes M) \oplus A(M \otimes M)$.*

Proof. Let T be as in Definition 5.4 then $\forall \mathbf{m} \in M \otimes M$ and $b \in \mathbb{C}[G]$ we have:

$$\begin{aligned} bT\left(\sum_{i,j} f_i c_j (\mathbf{m}_i \otimes \mathbf{m}_j)\right) &= \sum_k b_k g_k \sum_{i,j} f_i c_j (\mathbf{m}_j \otimes \mathbf{m}_i) \\ &= \sum_{i,j,k} b_k f_i c_j g_k (\mathbf{m}_j \otimes \mathbf{n}_i) \\ &= T\left(\sum_{i,j,k} b_k f_i c_j g_k (\mathbf{m}_i \otimes \mathbf{n}_j)\right) \\ &= Tb\left(\sum_{i,j} f_i c_j (\mathbf{m}_i \otimes \mathbf{m}_j)\right). \end{aligned}$$

Hence, $T : M \otimes M \to M \otimes M$ is a $\mathbb{C}[G]$-homomorphism. Let, $\mathbf{s} \in S(M \otimes M)$, $\mathbf{a} \in A(M \otimes M)$ and $b \in \mathbb{C}[G]$ then $T(b\mathbf{s}) = b\mathbf{s}$ and $T(b\mathbf{a}) = -b\mathbf{a}$. Therefore, $b\mathbf{s} \in S(M \otimes M)$ and $b\mathbf{a} \in A(M \otimes M)$, hence, $S(M \otimes M)$ and $A(M \otimes M)$ are $\mathbb{C}[G]$-submodules of $M \otimes M$.

If $\mathbf{m}$ is in the intersection of $S(M \otimes M)$ and $A(M \otimes M)$ then $\mathbf{m} = -\mathbf{m}$, hence, $\mathbf{m} = \mathbf{0}$. It follows that $M \otimes M$ decomposes into $S(M \otimes M) \oplus A(M \otimes M)$. □

Corollary 5.6. *Let M be a $\mathbb{C}[G]$-module affording the character χ so the character of $M \otimes M$ is χ^2. Let $S(M \otimes M)$ afford the character χ_S and $A(M \otimes M)$ afford the character χ_A then $\chi^2 = \chi_S + \chi_A$.*

We will usually refer to χ_S and χ_A as the symmetric and antisymmetric decomposition of χ^2.

Lemma 5.7. [JL01, p. 197] *Let $\mathbf{m}_1, \ldots, \mathbf{m}_n$ be a basis for M; then elements of the form $\mathbf{m}_i \otimes \mathbf{m}_j + \mathbf{m}_j \otimes \mathbf{m}_i$ for $1 \leq i \leq j \leq n$ are a $\frac{1}{2}n(n+1)$ dimensional basis for $S(M \otimes M)$ and elements of the form $\mathbf{m}_i \otimes \mathbf{m}_j - \mathbf{m}_j \otimes \mathbf{m}_i$ for $1 \leq i < j \leq n$ are a $\frac{1}{2}n(n-1)$ dimensional basis for $A(M \otimes M)$.*

Proof. Clearly, elements of the two forms are linearly independent. Moreover, $\dim(S(M \otimes M)) \geq \frac{1}{2}n(n+1)$ and $\dim(A(M \otimes M)) \geq \frac{1}{2}n(n-1)$. But $\dim(M \otimes M) = \dim(S(M \otimes M)) + \dim(A(M \otimes M)) = n^2$. Hence result. □

Theorem 5.8. [JL01, p. 198] *Let M be a $\mathbb{C}[G]$-module affording character χ. Let $M \otimes M$ have symmetric and antisymmetric modules $S(M \otimes M)$ and $A(M \otimes M)$, then for $g \in G$ we have $\chi_S(g) = \frac{1}{2}\left(\chi^2(g) + \chi(g^2)\right)$ and $\chi_A(g) = \frac{1}{2}\left(\chi^2(g) - \chi(g^2)\right)$.*

Proof. By Theorem 4.12, we can pick a basis of M such that $g(\mathbf{m}_i \otimes \mathbf{m}_j - \mathbf{m}_j \otimes \mathbf{m}_i) = \omega_i\omega_j(\mathbf{m}_i \otimes \mathbf{m}_j - \mathbf{m}_j \otimes \mathbf{m}_i)$ for complex numbers ω_i and ω_j. Hence, $\chi_A(g) = \sum_{i<j} \omega_i\omega_j$. Now, as $g^2(\mathbf{m}_i) = \omega_i^2\mathbf{m}_i$, it follows that $\chi(g) = \sum_i \omega_i$ and $\chi(g^2) = \sum_i \omega_i^2$. Observe that

$$\chi^2(g) = \left(\sum_i \omega_i\right)^2 = \sum_i \omega_i^2 + 2\sum_{i<j} \omega_i\omega_j = \chi(g^2) + 2\chi_A(g).$$

Hence, $\chi_A = \frac{1}{2}(\chi^2(g) - \chi(g^2))$. As $\chi^2 = \chi_S + \chi_A$ the result follows. □

Example 30. *The Character Table of $Sym(4)$*

Recall from Example 7 that $Sym(4)$ has 5 conjugacy classes with permutation cycle types 1^4, $1^2 2^1$, $1^1 3^1$ and 4^1. We can construct the trivial, sign and permutation characters denoted χ_0, χ_1 and χ_4 respectively. Note that each of these are irreducible.

$Sym(4)$	1^4	$1^2 2^1$	2^2	$1^1 3^1$	4^1
χ_0	1	1	1	1	1
χ_1	1	-1	1	1	-1
χ_4	3	1	-1	0	-1

Let χ_S and χ_A denote the symmetric and antisymmetric decomposition of χ_4^2 then we have:

χ_S	6	2	2	0	0
χ_A	3	-1	-1	0	1
χ_2	2	0	2	-1	0
χ_3	3	-1	-1	0	1

A quick calculation gives $\langle \chi_S, \chi_S \rangle = 3$ and $\langle \chi_A, \chi_A \rangle = 1$ therefore χ_A is irreducible. Checking the inner product of χ_S with each known irreducible we find that $\langle \chi_S, \chi_0 \rangle = 1$ and $\langle \chi_S, \chi_1\chi_A \rangle = 1$. Let $\chi_A = \chi_3$ and $\chi_2 = \chi_S - \chi_0 - \chi_3$ and note that $\langle \chi_2, \chi_2 \rangle = 1$. Table 5.1 shows the full character table of $Sym(4)$.

$Sym(4)$	1^4	$1^2 2^1$	2^2	$1^1 3^1$	4^1
χ_0	1	1	1	1	1
χ_1	1	-1	1	1	-1
χ_2	2	0	2	-1	0
χ_3	3	-1	-1	0	1
χ_4	3	1	-1	0	-1

Table 5.1: The character table of $Sym(4)$.

5.2 Restricting and Inducing Characters

Let $H \leq G$, then if M is a $\mathbb{C}[G]$-module, M is also a $\mathbb{C}[H]$-module. It follows that if M affords a character χ of G then χ is also a character of H; we denote this character of H as $\chi \downarrow H$. The character takes the value $\chi(h)$ for all $h \in H$ and we term it the *restriction of* χ *to* H.

If $\langle \cdot, \cdot \rangle_G$ is the usual class function inner product for a group G, then $\langle \cdot, \cdot \rangle_H$ will denote the usual class function inner product but with respect to the group H.

Example 31. *The Character Table of Alt*(4)
As an example of restriction, we will calculate the character table of $Alt(4)$. The conjugacy classes of $Alt(4)$ are the trivial class, a class of elements of cycle type 2^2, and two classes of elements of cycle type $1^1 3^1$.

We immediately have the trivial character and as $C_2^2 \triangleleft Alt(4)$ we can lift two characters from $Alt(4)/C_2^2$. We will denote these χ_0, χ_1 and χ_2 respectively.

$Alt(4)$	1^4	2^2	$1^1 3^1_A$	$1^1 3^1_B$
χ_0	1	1	1	1
χ_1	1	1	ω	ω^2
χ_2	1	1	ω^2	ω

Finally, we shall restrict χ_4 one of the 3 dimensional characters from $Sym(4)$ to $Alt(4)$. Let $\chi_3 = (\chi_4 \downarrow Alt(4))$.

χ_3	3	-1	0	0

We now observe that $\langle \chi_3, \chi_3 \rangle_{Alt(4)} = 1$, hence, χ_3 is irreducible. The full character table is given in Table 5.2.

$Alt(4)$	1^4	2^2	$1^1 3^1_A$	$1^1 3^1_B$
χ_0	1	1	1	1
χ_1	1	1	ω	ω^2
χ_2	1	1	ω^2	ω
χ_3	3	-1	0	0

Table 5.2: The character table of $Alt(4)$, where $\omega = e^{\frac{2i\pi}{3}}$.

Just as we can restrict characters to subgroups, it is equally possible to *induce* characters from a subgroup H to an overgroup G. If χ is a character of H then the *induction of* χ *to* G is denoted $\chi \uparrow G$.

Definition 5.9. [Isa94, p. 62] *Induced Class Function*
Let $H \leq G$ and let χ be a class function of H; then the *induced class function* on G is given by:

$$(\chi \uparrow G)(g) = \frac{1}{|H|} \sum_{x \in G} \dot{\chi}(xgx^{-1}) \quad \text{where} \quad \dot{\chi}(g) = \begin{cases} \chi(g) & \text{for } g \in H; \\ 0 & \text{for } g \notin H. \end{cases}$$

Theorem 5.10. [Isa94, p. 62] Frobenius Reciprocity Theorem
Let $H \leq G$ and suppose ϕ is a class function on H and ψ is a class function on G; then

$$\langle \phi, \psi \downarrow H \rangle_H = \langle \phi \uparrow G, \psi \rangle_G.$$

Proof.

$$\begin{aligned} \langle \phi \uparrow G, \psi \rangle_G &= \frac{1}{|G|} \sum_{g \in G} (\phi \uparrow G)(g) \overline{\psi}(g) \\ &= \frac{1}{|G| \times |H|} \sum_{g \in G} \sum_{x \in G} \dot{\phi}(xgx^{-1}) \overline{\psi}(g). \end{aligned}$$

Set $y = xgx^{-1}$ and note that $\psi(g) = \psi(y)$ then

$$\begin{aligned} \langle \phi \uparrow G, \psi \rangle_G &= \frac{1}{|G| \times |H|} \sum_{y \in G} \sum_{x \in G} \dot{\phi}(y) \overline{\psi}(g) \\ &= \frac{1}{|H|} \sum_{y \in H} \phi(y) \overline{\psi}(y) \\ &= \langle \phi, \psi \downarrow H \rangle_H. \end{aligned}$$

□

Definition 5.11. [JL01, p. 235] We define the *characteristic conjugacy class function* on a finite group G for an element x as

$$\zeta_x^G(g) = \begin{cases} 1 & \text{if } g \in x^G; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.12. [JL01, p. 235] *If χ is a character of G and x is an element of G then*

$$\langle \chi, \zeta_x^G \rangle_G = \frac{\chi(x)}{|C_G(x)|}.$$

Proof.

$$\langle \chi, \zeta_x^G \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi(g) \zeta_x^G(g) = \frac{1}{|G|} \sum_{g \in x^G} \chi(g) = \frac{\chi(x)}{|C_G(x)|}.$$

□

Theorem 5.13. [JL01, p. 236] *Let G be a finite group, $x \in G$ and $H \leq G$. Recall that $H \cap x^G$ breaks up into l conjugacy classes of H. Let χ be a character of H then:*

$$(\chi \uparrow G)(x) = |C_G(x)| \left(\frac{\chi(x_1)}{|C_H(x_1)|} + \cdots + \frac{\chi(x_l)}{|C_H(x_l)|} \right) \tag{5.2}$$

where $x_1, \ldots, x_l \in H$ are representative elements of the l classes of H.

Proof. Assume that x^G breaks up into l conjugacy classes in H then, $\zeta_x^G \downarrow H = \zeta_{x_1}^H + \cdots + \zeta_{x_l}^H$. By Theorem 5.10 and Lemma 5.12 we have:

$$\begin{aligned}
\frac{(\chi \uparrow G)(x)}{|C_G(x)|} &= \langle \chi, \zeta_x^G \downarrow H \rangle_H \\
&= \langle \chi, \zeta_{x_1}^H + \cdots + \zeta_{x_l}^H \rangle \\
&= \langle \chi, \zeta_{x_1}^H \rangle + \cdots + \langle \chi, \zeta_{x_l}^H \rangle \\
&= \frac{\chi(x_1)}{|C_H(x_1)|} + \cdots + \frac{\chi(x_l)}{|C_H(x_l)|}
\end{aligned}$$

The result follows. □

We will end this section with 3 examples; the first is a calculation of the character table of $Alt(5)$, which we will use in the second example. The second example will be fundamental to calculating the character tables of the small Mathieu groups. The final example is motivated by Example 12.

Example 32. *The Character Table of* $Alt(5)$
Let $G = Alt(5)$, then G has 5 conjugacy classes. We tabulate the orders of the classes in $Alt(5)$ and the centralisers in both $H = Alt(4) < Alt(5)$ and $Alt(5)$.

Cycle	1^5	$1^1 2^2$	$1^2 3^1$	5_A^1	5_B^1
Order	1	15	20	12	12
$\lvert C_G(g)\rvert$	60	4	3	5	5
$\lvert C_H(g)\rvert$	12	4	3	–	–

Note that although the class of 5 cycles has split from $Sym(5)$, an element g in one of these classes is conjugate to g^{-1}, but not to g^2 or g^3.

We immediately obtain two irreducible characters of $Alt(5)$, the trivial character χ_0 and the permutation character χ_3. The permutation character is given by the fact that $Alt(5)$ acts 3-transitively on 5 points and applying Theorem 4.37.

$Alt(5)$	1^5	2^2	3^1	5^1_A	5^1_B
χ_0	1	1	1	1	1
χ_3	4	0	1	-1	-1

Let χ be a character of $Alt(4)$ and let $g_A \in 1^1 3^1_A$ and $g_B \in 1^1 3^1_B$; then by (5.2) we have

$$(\chi \uparrow Alt(5)) = \begin{cases} 5\chi(g) & \text{if } g \in 1^5; \\ \chi(g) & \text{if } g \in 1^1 2^2; \\ \chi(g_A) + \chi(g_B) & \text{if } g \in 1^2 3^1; \\ 0 & \text{otherwise.} \end{cases}$$

If we induce the trivial character of $Alt(4)$ we obtain a character equal to $\chi_0 + \chi_3$. However, inducing either of the complex 1-dimensional characters of $Alt(4)$ gives a new irreducible character of $Alt(5)$, which we will denote χ_4. The remaining two characters can be calculated using the Schur orthogonality relations. The full character table is given in Table 5.3.

$Alt(5)$	1^5	2^2	3^1	5^1_A	5^1_B
χ_0	1	1	1	1	1
χ_1	3	-1	0	α	β
χ_2	3	-1	0	β	α
χ_3	4	0	1	-1	-1
χ_4	5	1	-1	0	0

Table 5.3: The character table of $Alt(5)$, where $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $\beta = \frac{1}{2}(1-\sqrt{5})$.

Example 33. *The Character Table of $Alt(6)$*
Let $G = Alt(6)$, then G has 7 conjugacy classes; these can be determined using the splitting criteria and are given below.

Cycle	1^6	2^2	3^1	3^2	$2^1 4^1$	5^1_A	5^1_B
Size	1	45	40	40	90	72	72
$\|C_G(g)\|$	360	8	9	9	4	9	9

As $Alt(6)$ acts 4-transitively on 6 points, by Theorem 4.37 points we immediately get a non trivial irreducible character, which we will denote χ_1.

$Alt(6)$	1^6	2^2	3^1	3^2	$2^1 4^1$	5^1_A	5^1_B
χ_0	1	1	1	1	1	1	1
χ_1	5	1	2	-1	-1	0	0

Let χ_S and χ_A be the symmetric and anti-symmetric decomposition of χ_1^2, the character values are given below.

χ_S	15	3	3	0	-1	0	0
χ_A	10	-2	1	1	0	0	0

We find that $\langle \chi_S, \chi_S \rangle = 3$, $\langle \chi_S, \chi_0 \rangle = 1$ and $\langle \chi_S, \chi_1 \rangle = 1$. Define $\chi_5 = \chi_S - \chi_0 - \chi_1$, and note that χ_5 is irreducible. Using the inner product again we find $\langle \chi_A, \chi_A \rangle = 1$; let $\chi_6 = \chi_A$, then χ_6 is irreducible.

We will now try inducing characters from $Alt(5)$ to $Alt(6)$. Let χ be a character of $Alt(5)$, then we can use the centraliser orders from the previous example to obtain the following equation.

$$(\chi \uparrow Alt(6)) = \begin{cases} 6\chi(g) & \text{if } g \in 1^5; \\ 2\chi(g) & \text{if } g \in 1^1 2^2; \\ 3\chi(g) & \text{if } g \in 1^2 3^1; \\ \chi(g) & \text{if } g \in 5^1_A, 5^1_B; \\ 0 & \text{otherwise.} \end{cases}$$

Inducing the trivial character of $Alt(5)$ gives a character equal to $\chi_0 + \chi_1$. Let χ_A and χ_B be the induced 3-dimensional characters. We find that $\langle \chi_A, \chi_A \rangle = \langle \chi_B, \chi_B \rangle = 2$ and χ_6 is a constituent of both χ_A and χ_B. Let $\chi_3 = \chi_A - \chi_6$ and $\chi_4 = \chi_B - \chi_6$; then both χ_3 and χ_4 are irreducible. Here, $\alpha = \frac{1}{2}(1 + \sqrt{5})$ and $\beta = \frac{1}{2}(1 - \sqrt{5})$.

χ_A	18	-2	0	0	0	α	β
χ_B	18	-2	0	0	0	β	α
χ_3	8	0	-1	-1	0	α	β
χ_4	8	0	-1	-1	0	β	α

The final character can easily be calculated using the Schur orthogonality relations. The full table is given in Table 5.4.

$Alt(6)$	1^6	2^2	3^1	3^2	$2^1 4^1$	5^1_A	5^1_B
χ_0	1	1	1	1	1	1	1
χ_1	5	1	2	-1	-1	0	0
χ_3	8	0	-1	-1	0	α	β
χ_4	8	0	-1	-1	0	β	α
χ_5	9	1	0	0	-1	-1	-1
χ_6	10	-2	1	1	0	0	0

Table 5.4: The character table of $Alt(6)$, where $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $\beta = \frac{1}{2}(1-\sqrt{5})$.

Example 34. *The Character Table of* $\mathrm{SL}_2(3)$
We will use several techniques including induction to construct the character table of $\mathrm{SL}_2(3)$. Recall that $\mathrm{SL}_2(3)$ is the set of invertible 2×2 matrices, with determinant equal to 1 and with entries in $\mathbb{F}_3$. We constructed this group in Example 12. The conjugacy classes can be determined by direct calculation and are given in Table 5.5.

Name	$C1$	$C2$	$C3_A$	$C3_B$	$C4$	$C6_A$	$C6_B$
Representative	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}2&0\\0&2\end{bmatrix}$	$\begin{bmatrix}1&1\\0&1\end{bmatrix}$	$\begin{bmatrix}1&2\\0&1\end{bmatrix}$	$\begin{bmatrix}0&2\\1&0\end{bmatrix}$	$\begin{bmatrix}2&1\\0&2\end{bmatrix}$	$\begin{bmatrix}2&2\\0&2\end{bmatrix}$
Order	1	1	4	4	6	4	4
Centraliser	24	24	6	6	4	6	6

Table 5.5: The conjugacy classes of $\mathrm{SL}_2(3)$.

First, observe that the union of the conjugacy classes $C1$, $C2$ and $C4$ is the group Q_8. Moreover, there are three cosets of $\mathrm{SL}_2(3)/Q_8$, namely $C3_A \cup C6_B$ and $C3_B \cup C6_A$. In fact, Q_8 is a normal subgroup and $\mathrm{SL}_2(3)/Q_8 \cong C_3$. We immediately get three characters, the trivial character χ_1 and two characters lifted from C_3. Let $\omega = e^{\frac{i\pi}{3}}$.

$\mathrm{SL}_2(3)$	$C1$	$C2$	$C3_A$	$C3_B$	$C4$	$C6_A$	$C6_B$
χ_0	1	1	1	1	1	1	1
χ_1	1	1	ω	ω^2	1	ω	ω^2
χ_2	1	1	ω^2	ω	1	ω^2	ω

To obtain more characters we will attempt to induce characters from Q_8 to $\mathrm{SL}_2(3)$. The following table shows how Q_8 exists inside $\mathrm{SL}_2(3)$ and lists the order of the centraliser of each element.

Class in Q_8	$C1$	$C2$	$C4_A$	$C4_B$	$C4_C$
$\lvert C_{Q_8}(g)\rvert$	8	8	4	4	4
$\lvert C_{\mathrm{SL}_2(3)}(g)\rvert$	24	24	4		

Let χ be a character of Q_8 and let $g_A \in C4_A$, $g_B \in C4_B$ and $g_C \in C4_C$. Then:

$$(\chi \uparrow \mathrm{SL}_2(3))(g) = \begin{cases} 3\chi(g) & \text{if } g \in C1, C_2; \\ \chi(g_A) + \chi(g_B) + \chi(g_C) & \text{if } g \in C_4; \\ 0 & \text{otherwise.} \end{cases}$$

Let χ_6 and χ_X be the induced trivial character, and the induced 2-dimensional character of Q_8 to $\mathrm{SL}_2(3)$ respectively. Then we obtain:

χ_6	3	3	0	0	-1	0	0
χ_X	6	-6	0	0	0	0	0

As $\langle \chi_6, \chi_6 \rangle = 1$, we see that χ_6 is irreducible. But, $\langle \chi_X, \chi_X \rangle = 3$ and it is not orthogonal to any known irreducible characters.

We will now attempt to induce characters from another subgroup isomorphic to C_3. The centraliser of every element in C_3 is of size 3, whereas for order three elements in $\mathrm{SL}_2(3)$ it is of size 6. Now if χ is a character of C_3, then:

$$(\chi \uparrow \mathrm{SL}_2(3))(g) = \begin{cases} 8\chi(g) & \text{if } g \in C1; \\ 2\chi(g) & \text{if } g \in C3_A, C3_B; \\ 0 & \text{otherwise.} \end{cases}$$

Let χ_Z be the induced trivial character from C_3 to $\mathrm{SL}_2(3)$. We have $\langle \chi_Z, \chi_Z \rangle = 4$, $\langle \chi_Z, \chi_0 \rangle = 1$ and $\langle \chi_Z, \chi_6 \rangle = 1$. Let $\chi_Y = \chi_Z - \chi_0 - \chi_6$.

χ_Z	8	0	2	2	0	0	0
χ_Y	4	-4	1	1	0	-1	-1

We see that $\langle \chi_X, \chi_Y \rangle = 2$. Now let $\chi_4 = \chi_X - \chi_Y$, then $\langle \chi_4, \chi_4 \rangle = 1$. Hence, χ_4 is irreducible. Moreover, taking the tensor product of χ_4 with either χ_1 or χ_2 gives a new irreducible character. Table 5.6 shows the full character table.

$\mathrm{SL}_2(3)$	$C1$	$C2$	$C3_A$	$C3_B$	$C4$	$C6_A$	$C6_B$
χ_0	1	1	1	1	1	1	1
χ_1	1	1	ω^2	ω^4	1	ω^2	ω^4
χ_2	1	1	ω^4	ω^2	1	ω^4	ω^2
χ_3	2	-2	-1	-1	0	1	1
χ_4	2	-2	ω^5	ω	0	ω^2	ω^4
χ_5	2	-2	ω	ω^5	0	ω^4	ω^2
χ_6	3	3	0	0	-1	0	0

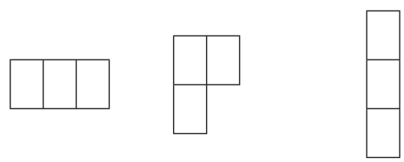
Table 5.6: The character table of $\mathrm{SL}_2(3)$, where $\omega = e^{\frac{i\pi}{3}}$.

5.3 Characters of the Symmetric Group

In this section we will outline the construction of character values for the symmetric group. We will not be proving results, our aim is simply to get to a usable formula. Recall that the conjugacy classes of $Sym(n)$ are indexed by *partitions* of n. We will denote a partition of $n = \lambda_1 + \cdots + \lambda_k$ as $\lambda = (\lambda_1, \ldots, \lambda_k)$ with each $\lambda_i \geq \lambda_{i+1}$.

Definition 5.14. [FH91, p. 45] *Young Diagram*
Given a λ a partition of n, the *Young Diagram* corresponding to λ is a left justified array of boxes, with row i having λ_i boxes.

Example 35. The Young diagrams corresponding to the partitions of 3 are:



$\lambda = (3) \quad \lambda = (2,1) \quad \lambda = (1,1,1)$

We now define a λ-*tableau* to be a numbering of a Young diagram with distinct numbers from 1 to n. Given a λ-*tableau* t, we note that $Sym(n)$ acts naturally on the set of symbols. We define:

$$R(t) = \{g \in Sym(n) \colon g \text{ preserves each row of } t\};$$

$$C(t) = \{g \in Sym(n) \colon g \text{ preserves each column of } t\}.$$

Example 36. Let $\lambda = (4, 3, 1)$ and let

$$t_\lambda = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 8 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

We see that $Sym(\{1,2,3,4\})$, $Sym(\{5,6,7\})$ and $Sym(\{8\})$ preserve every row. Hence, $R(t) = Sym(\{1,2,3,4\}) \times Sym(\{5,6,7\}) \times Sym(\{8\})$. We also see that $Sym(\{1,5,8\})$, $Sym(\{2,6\})$, $Sym(\{3,7\})$ and $Sym(\{4\})$ preserve every column. Hence, $C(t) = Sym(\{1,5,8\}) \times Sym(\{2,6\}) \times Sym(\{3,7\}) \times Sym(\{4\})$.

Definition 5.15. [JK81, p. 41] λ-*Tabloid*
A λ-*tabloid*, denoted $\{t\}$, is the equivalence class of a λ-tableau t under the action of $C(t)$.

We claim that the action of $Sym(n)$ on the tableaux extends to the set of λ-tabloids. Let $\{t\}$ be a λ-tabloid, then we extend this action by defining $\forall g \in Sym(n)$, $g\{t\} = \{gt\}$. Now, let M^λ be a $\mathbb{C}[Sym(n)]$-module spanned by the $\mathbb{C}$-linear combinations of tabloids. In general this module is not *irreducible*, however a submodule of it is, which we are close to defining.

Definition 5.16. [JK81, p. 295] *λ-Polytabloid*
We define the *λ-polytabloid*, denoted $e(t)$, of a λ-tableau t as:

$$e(t) = \{t\} \sum_{g \in C(t)} sgn(g)g.$$

Here, $sgn : Sym(n) \to \{1, -1\}$ is the usual sign homomorphism for permutation groups.

Definition 5.17. [JK81, p. 296] *(Complex) Specht Module*
Let λ be a partition of n. Let M^λ be a $\mathbb{C}[Sym(n)]$-module spanned by the $\mathbb{C}$-linear combinations of tabloids. We define the *(complex) Specht module* S^λ to be the subspace of M^λ spanned by the λ-polytabloids. We denote the character of a Specht module by χ_λ.

Theorem 5.18. [JK81, p. 298] *Let λ be a partition of n; then the (complex) Specht module S^λ is irreducible and uniquely determined by λ.*

We omit the proof of this theorem, however, a proof and a full exposition of the ideas behind the proof can be found in James's 'The Representation Theory of the Symmetric Groups' [Jam78]. We are almost ready to state the Frobenius character formula. The following result is taken from [FH91], a proof can be found there as well.

Definition 5.19. [FH91, p. 48] *The Frobenius Character Formula*
Let λ be a partition of n. Introduce independent variables $x_1, \dots, x_k$ with k greater than or equal to the number of rows of the Young diagram. Define the following polynomials:

$$P_j(x) = x_1^j + \cdots + x_k^j;$$

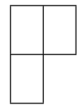
$$\Delta(x) = \prod_{i<j}(x_i - x_j).$$

Recall that $\lambda = (\lambda_1, \dots, \lambda_k)$ and set $l_i = \lambda_i + k - i$. We also let the partition $C_i = (i_1, \dots, i_n)$ represent a conjugacy class of $Sym(n)$. Here, each i_j denotes the number of j-cycles in an element of the class. The coefficient of $x_1^{l_1} \dots x_k^{l_k}$ in

$$\Delta(x) \prod_{j=1}^{n} P_j(x)^{i_j} \tag{5.3}$$

is exactly $\chi_\lambda(g)$ for $g \in C_i$.

Example 37. Let $n = 3$ and let $\lambda = (2, 1)$; this corresponds to the Young diagram:



We now introduce variables x_1 and x_2. Using $l_i = \lambda_i + k - i$, we have $l_1 = 3$ and $l_2 = 1$, therefore we are looking at coefficients of $x_1^3 x_2$. As there are two variables we see that $\Delta(x) = (x_1 - x_2)$.

C_i	Polynomial	Coefficient of $x_1^3 x_2$
$(3, 0, 0)$	$(x_1 - x_2)(x_1 + x_2)^3 = x_1^4 + 2x_1^3 x_2 + \ldots$	2
$(1, 1, 0)$	$(x_1 - x_2)(x_1 + x_2)(x_1^2 + x_2)^2 = x_1^4 - x_2^4$	0
$(0, 0, 1)$	$(x_1 - x_2)(x_1^3 + x_2^3) = x_1^4 - x_1^3 x_2 + \ldots$	-1

Given that $(3, 0, 0)$ is the identity, $(1, 1, 0)$ is the class of transpositions and $(0, 0, 1)$ is the class of 3-cycles; we have just calculated the 2-dimensional character of $Sym(3)$. This result can be compared with Example 26.

Chapter 6

Constructing the Small Mathieu Groups

In this chapter we will construct the small Mathieu groups as a series of transitive extensions of multiply transitive permutation groups. To do this we will apply Theorem 6.2 to a non split extension of $Alt(6)$ to obtain M_{11}. Applying the same theorem to M_{11} will construct M_{12}.

6.1 Extending Multiply Transitive Groups

Theorem 6.1. [BW79, p .10] *Let G act transitively on a set Δ and let $\delta \in \Delta$. Then G is k-transitive if and only if G_δ acts $(k-1)$-transitively on $\Delta - \{\delta\}$. Moreover, if G is k-transitive on Δ and $k \geq 2$, then the following hold:*

1. *$G = G_\delta \cup G_\delta g G_\delta$ for any $g \in G$ such that $g \notin G_\delta$;*

2. *G_δ is a maximal subgroup of G.*

Proof. Suppose that G_δ is $(k-1)$-transitive on $\Delta - \{\delta\}$ and let $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta = (\beta_1, \ldots, \beta_k) \in \Delta^{(k)}$. As G is transitive we can select elements g_1, g_2 such that $g_1(\alpha_1) = \delta$ and $g_2(\beta_1) = \delta$. As G_δ is $(k-1)$-transitive we can choose $h \in G_\delta$ such that $h(g_1(\alpha_i)) = g_2(\beta_i)$ for $i = 2, \ldots, k$. The element given by $g_2^{-1} h g_1 \in G$ sends α to β, hence, G is k-transitive.

For the converse, let $(\alpha_1, \ldots, \alpha_k), (\beta_1, \ldots, \beta_k) \in \Delta^{(k)}$. As G acts transitively, there exists $g, h \in G$ such that $g\alpha_1 = h\beta_1 = \delta$. Now, observe that $(g\alpha_2, \ldots, g\alpha_k)$ and $(h\beta_2, \ldots, h\beta_k)$ are elements of $(\Delta - \{\delta\})^{(k-1)}$. Thus, there exists $k \in G_\delta$ such that $k(g\alpha_i) = \beta_i$ for $i = 2, \ldots, k$. Now, $h^{-1}(kg\alpha_i) = h^{-1}(h\beta_i) = \beta_i$ for all $i = 1, \ldots, k$. Therefore, G_δ acts $(k-1)$-transitively on $\Delta - \{\delta\}$.

To prove (1), consider an element $h \in G$ such that $h \notin G_\delta$. Now, let $g_1 \in G_\delta$ and $g_2 \in G$ such that $g_1(g_2^{-1}(\delta)) = h^{-1}(\delta)$, then $hg_1g_2^{-1} \in G_\delta$. It follows that $h \in G_\delta g_2 \delta$, hence, $G = G_\delta \cup G_\delta g G_\delta$.

To prove (2), assume that that there exists a subgroup H of G such that $G \geq H > G_\delta$. By (1), $G = G_\delta \cup G_\delta g G_\delta$ for any $g \in G$ such that $g \notin G_\delta$; pick a $g \in H - G_\delta$, then $G = G_\delta \cup G_\delta g G_\delta \subseteq H$ which implies $G = H$. Hence, G_δ is maximal in G. □

For the rest of this chapter we will construct a finite family of multiply transitive groups called the small Mathieu groups, denoted M_9, M_{10}, M_{11} and M_{12}. We will later discover that M_{11} and M_{12} are 2 of the 26 sporadic simple groups. The construction we will use follows a series of exercises from 'Permutation Groups and Combinatorial Structures' by Biggs and White [BW79]. The following theorem will be fundamental to the construction of the Mathieu groups.

Theorem 6.2. [BW79, p .12] *Let G be a group acting k-transitively on Δ with $k \geq 2$. Let, $\Delta^\dagger = \Delta \cup \{\star\}$ and suppose we can find a permutation h of $\Delta^\dagger$ and a $g \in G$ such that*

1. *h switches $\star$ and some point $\delta \in \Delta$ and fixes some point $\omega \in \Delta$;*
2. *g switches δ and ω;*
3. *h^2 and $(gh)^3$ are in G*
4. *$hG_\delta h = G_\delta$.*

If 1 to 4 hold then $G^\dagger = \langle G, h\rangle$ acts on $\Delta^\dagger$ $(k+1)$-transitively.

Proof. By Theorem 6.1 we have $G = G_\delta \cup G_\delta g G_\delta$, we will now show that $\langle G, h\rangle = G \cup GhG$. First, we will show that $G \cup GhG$ is a group. We only need to show closure; since $G \cup GhG$ is closed under inverses ($h^2 \in G$), it is sufficient to check that $hGh \subseteq G \cup GhG$. Observe that h^2 fixes δ and $h^2 \in G$ so $h^2 \in G_\delta$, hence, by item 4 we have $hG_\delta = G_\delta h$. As $(gh)^3 \in G$, we have $hgh \in (ghg)^{-1}G = g^{-1}h^{-1}G = g^{-1}hG$. Now we can show that $hGh \subseteq G \cup GhG$;

$$\begin{aligned} hGh &= h(G_\delta \cup G_\delta g G_\delta)h \\ &= hG_\delta h \cup hG_\delta g G_\delta g G_\delta h \\ &= G_\delta \cup G_\delta \cdot hgh \cdot G_\delta \\ &\subseteq G \cup G_\delta \cdot g^{-1}hG \cdot G_\delta \\ &\subseteq G \cup GhG. \end{aligned}$$

It follows that $GhG \cdot GhG = G \cdot hGh \cdot G \subseteq G \cup GhG$, hence, $G \cup GhG = \langle G, h \rangle$. As nothing in GhG fixes $\star$ it follows that $(G^\dagger)_\star = G$. Moreover, $G^\dagger$ is $(k+1)$-transitive on $\Delta^\dagger$ by Theorem 6.1. □

6.2 Constructing M_9 and M_{10}

We note that all proofs throughout this construction are original work guided by the exercises in [BW79]. In addition, any explanatory notes, Theorem 6.8 and Theorem 6.10 are completely original work.

Let G be a group acting on a set Δ, then G also acts on the set of subsets of Δ. Let $\Lambda \subseteq \Delta$, $\lambda \in \Lambda$ and $g \in G$. If, for every $\lambda \in \Lambda$ we have $g\lambda = \lambda$ then g fixes Λ *pointwise*. If, for every $\lambda \in \Lambda$ we have $g\lambda \in \Lambda$ then g fixes Λ *setwise*.

Let $\Delta = \{a, b, c, d, e, f\}$ and let $Sym(\Delta)$ act on Δ as an object in $\mathfrak{Set}$, it follows that $Sym(\Delta) = Aut_{\mathfrak{Set}}(\Delta)$. Now define $\Omega = \{0, \ldots, 9\}$ and identify each $\omega \in \Omega$ with one of the partitions of Δ into two sets of three.

0	$\{abc\}\{def\}$	5	$\{ace\}\{bdf\}$
1	$\{abd\}\{cef\}$	6	$\{acf\}\{bde\}$
2	$\{abe\}\{cdf\}$	7	$\{ade\}\{bcf\}$
3	$\{abf\}\{cde\}$	8	$\{adf\}\{bce\}$
4	$\{acd\}\{bcf\}$	9	$\{aef\}\{bcd\}$

Proposition 6.3. *Let $G = Sym(\Delta)$, then G acts transitively and faithfully on Ω.*

Proof. Consider G_0, the pointwise stabiliser of 0 in Ω. Observe that $\{a, b, c\}$ is fixed setwise by $Sym(\{a, b, c\})$ and $\{d, e, f\}$ is fixed setwise by $Sym(\{d, e, f\})$. Finally, 0 can be fixed by swapping $\{a, b, c\}$ with $\{d, e, f\}$. As all of $\{a, b, c\}$ must be swapped with all of $\{d, e, f\}$ we need a permutation that acts by swapping elements of $Sym(\{a, b, c\})$ with elements of $Sym(\{d, e, f\})$; one such permutation[1] is $(ae)(bf)(cd)$. It follows that[2] $G_0 = (Sym(3) \times Sym(3)) \rtimes C_2$ and $|G_0| = (6 \times 6) \times 2 = 72$. Now, $|G| = 720$ and $|\Omega| = 10$, by the Orbit Stabiliser Theorem (Theorem 2.9),

$$|0^G| = |G|/|G_0| = 720/72 = 10 = |\Omega.|$$

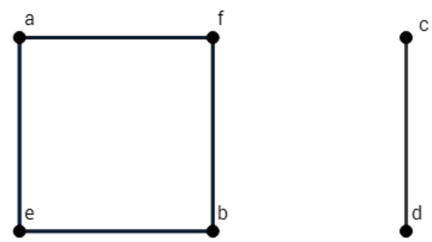
Hence, G acts transitively on Ω. Notice that the intersection of any five stabilisers of points in Ω is trivial, hence G acts faithfully on Ω. □

[1]Another is given by $(afbe)(cd)$.

[2]This semi-direct product is actually a wreath product $Sym(3) \wr Sym(2)$.

Proposition 6.4. *Let $G = Sym(\Delta)$, then G acts 2-transitively on Ω but not 3-transitively.*

Proof. The stabiliser of 0 and 1 is isomorphic to D_8; this can be observed by considering $G_{01} = \langle (afbe)(cd), (ab) \rangle$ and noting that $G_{01} < Aut_{\mathfrak{Graph}}(\Gamma)$ where:[3]

$\Gamma =$ 

It follows that $|G_{01}| = 8$. Now $|G| = 720$ and $|\Omega^{(2)}| = 90$, and by the Orbit Stabiliser Theorem we have,

$$|(0,1)^G| = 720/8 = 90 = |\Omega^{(2)}|. \tag{6.1}$$

Hence, G acts 2-transitively on Ω. On the other hand the point stabiliser G_{012} contains the permutation (ab), and so since $|\Omega^{(3)}| = 720$ and $|G|/|G_{012}| < 720$. It follows that G is not 3-transitive. □

Proposition 6.5. *Let $H = Alt(\Delta) < G$. Then H is 2-transitive on Ω, $H_{01} = \langle (afbe)(cd) \rangle \cong C_4$, $H_0 = \langle (afbe)(cd), (abc), (def) \rangle$ and $H = \langle H_0, \psi \rangle$ where $\psi \in H - H_0$.*

Proof. As H contains only the even permutations in G, the stabiliser H_{01} only contains the even permutations of G_{01}. The even permutations are exactly the elements generated by the order four permutation $(afbe)(cd)$, hence, $H_{01} = \langle (afbe)(cd) \rangle$. As H_{01} is generated by a single order 4 element $H_{01} \cong C_4$. The same argument applies to H_0 (recall that in $Sym(3)$ the only even permutations are the identity and the three cycles), and it immediately follows that $H_0 = \langle (afbe)(cd), (abc), (def) \rangle$.

To see that H is 2-transitive, observe that by only considering even permutations we have halved $|G|$ and $|G_{01}|$, therefore (6.1) remains unchanged and H is 2-transitive on Ω. By 2-transitivity H_0 is maximal in H, therefore adding another element $\psi \in H - H_0$ must generate H, hence $H = \langle H_0, \psi \rangle$. □

Proposition 6.6. *Let $\phi_1 = (abc) = (194)(285)(376)$, $\phi_2 = (def) = (123)(456)(798)$, $\theta = (afbe)(cd) = (2934)(5876)$ and $\psi = (ab)(cd) = (01)(49)(56)(78)$ so $H = \langle \phi_1, \phi_2, \theta, \psi \rangle$. Define $\lambda = (2735)(4698)$. Then $\lambda \notin H$, but $\lambda^2 \in H$ and λ is an outer automorphism of H. In particular H is a normal index 2 subgroup of $\langle H, \lambda \rangle$.*

[3] $Aut_{\mathfrak{Graph}}(\Gamma) \cong D_8 \times C_2$.

Proof. It is easy to see that $\lambda \notin H$ as $\lambda \notin H_{01}$ and λ stabilises both 0 and 1. As $\lambda^2 = \theta^2$ we see that $\lambda^2 \in H$. To show λ is an outer automorphism we will check that the conjugate of each generator of H by λ is in H.

$$\begin{aligned}
\lambda\phi_1\lambda^{-1} &= (2735)(4698)(194)(285)(376)(2537)(4896) &&= (186)(742)(539) &&= \phi_1\phi_2 \\
\lambda\phi_2\lambda^{-1} &= (2735)(4698)(123)(456)(798)(2537)(4896) &&= (175)(629)(384) &&= \phi_1\phi_2^{-1} \\
\lambda\theta\lambda^{-1} &= (2735)(4698)(2934)(5876)(2537)(4896) &&= (2439)(5678) &&= \theta^{-1} \\
\lambda\psi\lambda^{-1} &= (2735)(4698)(01)(49)(56)(78)(2537)(4896) &&= (01)(43)(29)(68) &&= \psi\theta^{-1}
\end{aligned}$$

Clearly, each conjugate is in H so $\lambda \in Aut_{\mathfrak{Grp}}(H)$; as $\lambda \notin H$ and λ does not centralise H, it follows that λ is an outer automorphism of H. As $\lambda^2 \in H$ it follows that $H = \lambda^2 H$, multiplying both sides by λ gives $\lambda H = \lambda^3 H$. Hence, there are two cosets of H in $\langle H, \lambda\rangle$ and $|\langle H, \lambda\rangle \colon H| = 2$. As H is an index 2 subgroup, it follows that $H \triangleleft \langle H, \lambda\rangle$. □

Definition 6.7. The Mathieu Group M_{10}
We define the *Mathieu group on 10 points* as $M_{10} = \langle H, \lambda\rangle = \langle \phi_1, \phi_2, \theta, \psi, \lambda\rangle$.

Theorem 6.8. *$M_{10} \cong Alt(6)^{\cdot}C_2$.*

We know that $|M_{10} : Alt(6)| = 2$, to show that M_{10} is a non split extension it suffices to show there are no order 2 elements in $\lambda Alt(6)$. If there were any order two elements in $\lambda Alt(6)$ then the following short exact sequence would split:

$$\{1\} \longrightarrow Alt(6) \longrightarrow M_{10} \longrightarrow C_2 \longrightarrow \{1\}.$$

Proof. Recall that $M_{10} = \langle \phi_1, \phi_2, \theta, \psi, \lambda\rangle$ and $Alt(6) = \langle \phi_1, \phi_2, \theta, \psi\rangle$. Let $Q = \langle \theta, \psi, \lambda\rangle$; then $Q \in Syl_2(M_{10})$. Let $P = \langle \psi, \theta\rangle$; then $P \in Syl_2(Alt(6))$. It follows that $|Q : P| = 2$ and $P \triangleleft Q$, moreover, $Q/P = \{P, \lambda P\}$. To check there are no order 2 elements in $\lambda Alt(6)$, we just need to check there are no order 2 elements in λP.

P			λP		
1			λ	=	(2735)(4698)
θ	=	(2934)(5876)	$\lambda\theta$	=	(2836)(4795)
θ^2	=	(23)(49)(57)(68)	$\lambda\theta^2$	=	(2537)(4896)
θ^3	=	(2439)(5678)	$\lambda\theta^3$	=	(2638)(4597)
ψ	=	(01)(49)(56)(78)	$\lambda\psi$	=	(01)(48359627)
$\theta\psi$	=	(01)(43)(68)(29)	$\lambda\theta\psi$	=	(01)(45289736)
$\theta^2\psi$	=	(01)(23)(58)(67)	$\lambda\theta^2\psi$	=	(01)(25463798)
$\theta^3\psi$	=	(01)(24)(39)(57)	$\lambda\theta^3\psi$	=	(01)(26953847)

As there are no order 2 elements in λP, there are no order 2 elements in $\lambda Alt(6)$ and we see that $M_{10} = Alt(6)^{\cdot}C2$. $\square$

Definition 6.9. The Mathieu Group M_9
Let $G = M_{10}$ then we define the *Mathieu group on 9 points* as $M_9 = G_0$. That is, M_9 is the point stabiliser of 0 in M_{10} acting on Ω.

Theorem 6.10. *Let $G = M_{10}$. Then $|G| = 720$, G is sharply 3-transitive on Ω, $G_{01} \cong Q_8$ and $G_0 \cong \mathbb{F}_3^2 \rtimes_\eta Q_8$ where Q_8 is acting via the faithful two-dimensional irreducible representation over $\mathbb{F}_3$.*

Proof. Recall that $G = \langle \phi_1, \phi_2, \theta, \psi, \lambda \rangle$. Let $H = \langle \phi_1, \phi_2, \theta, \psi, \lambda \rangle \cong Alt(6) < G$ and recall that $H_0 = \langle \phi_1, \phi_2, \theta \rangle$ and $H_{01} = \langle \theta \rangle$. As G is an extension of H by λ and λ fixes both 0 and 1, it follows that $G_0 = \langle \phi_1, \phi_2, \theta, \lambda \rangle$ and $G_{01} = \langle \theta, \lambda \rangle$. The only permutation that fixes three points is the identity, hence $G_{012} = \{1\}$. We can see that $|G| = 2 \times |Alt(6)| = 2 \times \frac{1}{2} \times 6! = 720$. Consider the orbit of $(0, 1, 2)$. We have $|(0, 1, 2)^G| = |G|/|G_{012}| = 720 = |\Omega^{(3)}|$. Clearly, G acts sharply 3-transitive on Ω.

Now, $G_{01} = \langle \theta, \lambda \rangle$, it is easy to $\theta^2 = \lambda^2 = (\theta\lambda)^2$ and $(\theta\lambda)^4 = 1$. Recall the presentation for Q_8 is $\langle i, j, k \colon i^2 = j^2 = k^2 = ijk,\ (ijk)^2 = 1 \rangle$. Now, set $i = \theta$ and $j = \lambda$; then $k = \theta\lambda$ and we see that $Q_8 \cong \langle \theta, \lambda \colon \theta^2 = \lambda^2 = (\theta\lambda)^2,\ (\theta\lambda)^4 = 1 \rangle$. It is immediate that $Q_8 \cong G_{01}$.

Consider ϕ_1 and ϕ_2. As ϕ_1, ϕ_2 and their product are order 3, and each of those elements commute it follows that $\langle \phi_1, \phi_2 \rangle \cong C_3^2$. Now define an action:

$$\varepsilon \colon \langle \theta, \lambda \rangle \times \langle \phi_1, \phi_2 \rangle \to \langle \phi_1, \phi_2 \rangle \text{ by } \varepsilon(g, \phi) \mapsto g\phi g^{-1}.$$

The triple $(\langle \theta, \lambda \rangle, \langle \phi_1, \phi_2 \rangle, \varepsilon)$ is an object in $\mathfrak{GSet}$. Define an embedding:

$$\xi_1 \colon \langle \theta, \lambda \rangle \to GL_3(2) \text{ by } \theta \mapsto \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \text{ and } \lambda \mapsto \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

This $Q_8 < GL_3(2)$ acts on $\mathbb{F}_3^2$ as an object in $\mathfrak{Vect}_{\mathbb{F}_3}$ by matrix multiplication (we saw this in Example 12). The triple $(Q_8, \mathbb{F}_3^2, \eta)$ is an object in $\mathfrak{GSet}$. Define a group isomorphism

$$\xi_2 \colon \langle \phi_1, \phi_2 \rangle \to \mathbb{F}_3^2 \text{ by } \phi_1 \mapsto [1\ 0]^\intercal \text{ and } \phi_2 \mapsto [0\ 1]^\intercal.$$

The pair $(\varepsilon_1, \varepsilon_2)$ is an arrow in $\mathfrak{GSet}$. Moreover, the following diagram commutes.

$$\begin{array}{ccc} \langle \theta, \lambda \rangle \times \langle \phi_1, \phi_2 \rangle & \xrightarrow{(\xi_1,\xi_2)} & Q_8 \times \mathbb{F}_3^2 \\ \Big\downarrow{\scriptstyle \varepsilon} & & \Big\downarrow{\scriptstyle \eta} \\ \langle \phi_1, \phi_2 \rangle & \xrightarrow[\xi_2]{} & \mathbb{F}_3^2 \end{array}$$

It follows that $\langle \phi_1, \phi_2, \theta, \lambda \rangle \cong \mathbb{F}_3^2 \rtimes_\eta Q_8$. □

6.3 Constructing M_{11} and M_{12}

Proposition 6.11. *Let $G = M_{10}$ and let $\Lambda = \Delta \cup \{X\}$ then $\mu = (0X)(47)(59)(68)$ and $\psi = (01)(49)(56)(78)$ satisfy the conditions of Theorem 6.2.*

Proof. To see (1) and (2), we observe that μ switches 0 and X but fixes 1, and that ψ switches 0 and 1. To see (3) we note that $\mu^2 = 1$ so $\mu^2 \in G$ and we shall check that $(\psi\mu)^3 \in G$:

$$\begin{aligned} (\psi\mu)^3 &= ((0X)(47)(59)(68)(01)(49)(56)(78))^3 \\ &= ((01X)(458)(697))^3 \\ &= 1. \end{aligned}$$

Hence, $(\psi\mu)^3 \in G$. Finally, we can verify (4) by calculating $\mu G_0 \mu$, as μ is order 2. We can just check the conjugates of the generators of G_0:

$$\begin{aligned} \mu\phi_1\mu &= \mu(194)(285)(376)\mu = (157)(269)(348) = \phi_1^2\phi_2; \\ \mu\phi_2\mu &= \mu(123)(456)(798)\mu = (123)(798)(456) = \phi_2; \\ \mu\theta\mu &= \mu(2934)(5876)\mu \quad = (2637)(9648) \quad = \lambda^{-1}; \\ \mu\lambda\mu &= \mu(2735)(4698)\mu \quad = (2439)(7856) \quad = \theta^{-1}. \end{aligned}$$

Hence, $\mu G_0 \mu = G_0$ and it follows that $\langle G, \mu \rangle$ is a transitive extension of G. □

Definition 6.12. The Mathieu Group M_{11}
We define the *Mathieu group on 11 points* as $M_{11} = \langle \phi_1, \phi_2, \theta, \psi, \lambda, \mu \rangle = \langle M_{10}, \mu \rangle$.

Theorem 6.13. *Let $G = M_{11}$. Then $|G| = 7920$, G is sharply 4-transitive on Λ and G is a proper subgroup of $Alt(11)$.*

Proof. Let $H = M_{10} = G_X$, then $G_{012X} = H_{012} = \{1\}$ and it follows that H is sharply 4-transitive. By the Orbit Stabiliser Theorem we see that $|G| = |\Lambda| \times |G_X| = 11 \times 720 = 7920$. As G is a permutation group on 11 points, is composed of only even permutations, and has order less than $\frac{1}{2} \times 11!$; it follows that $G < Alt(11)$. □

Proposition 6.14. *Let $G = M_{11}$ and $\Pi = \Lambda \cup \{\infty\}$, then $\sigma = (X\infty)(49)(58)(67)$ and $\mu = (0X)(47)(59)(68)$ satisfy the conditions of Theorem 6.2.*

Proof. To see 1 and 2 we observe that σ switches X and ∞ but fixes 0 and that μ switches 0 and X. To see 3 we note that $\sigma^2 = 1$ so $\sigma^2 \in G$ and we shall check $(\mu\sigma)^3 \in G$:

$$\begin{aligned}(\mu\sigma)^3 &= ((47)(59)(68)(0X)(49)(58)(67)(x\infty))^3 \\ &= ((456)(789)(0X\infty))^3 \\ &= 1.\end{aligned}$$

Finally we can verify 4 by calculating $\sigma G_X \sigma$. As σ is order 2 we can just check the conjugates of the generators of G_X.

$$\begin{aligned}\sigma\phi_1\sigma &= \sigma(194)(285)(376)\sigma &&= (149)(258)(367) &&= \phi_1^2; \\ \sigma\phi_2\sigma &= \sigma(123)(456)(798)\sigma &&= (123)(798)(456) &&= \phi_2; \\ \sigma\theta\sigma &= \sigma(2934)(5876)\sigma &&= (2439)(8567) &&= \theta^{-1}; \\ \sigma\lambda\sigma &= \sigma(2735)(4698)\sigma &&= (2638)(9745) &&= \theta\lambda; \\ \sigma\psi\sigma &= \sigma(01)(49)(87)(56)\sigma &&= (01)(49)(87)(56) &&= \psi.\end{aligned}$$

Hence, $\sigma G_X \sigma = G_X$ and it follows that $\langle G, \sigma\rangle$ is a transitive extension of G. □

Definition 6.15. The Mathieu Group M_{12}
We define the *Mathieu group on 12* points as $M_{12} = \langle \phi_1, \phi_2, \theta, \psi, \lambda, \mu, \sigma\rangle = \langle M_{11}, \sigma\rangle$.

Theorem 6.16. *Let $G = M_{12}$. Then $|G| = 95040$, G is sharply 5-transitive on Π and G is a proper subgroup of $Alt(12)$.*

Proof. Let $H = M_{11} = G_\infty$. Then $G_{012X\infty} = H_{012X} = \{1\}$ and it follows that G is sharply 5-transitive. By the Orbit Stabiliser Theorem we see that $|G| = |\Pi| \times |G_\infty| = 12 \times 7920 = 95040$. As G is a permutation group on 12 points, is composed of only even permutations and has order less than $\frac{1}{2} \times 12!$, it follows that $G < Alt(12)$. □

Chapter 7

Conjugacy Class Structure of the Small Mathieu Groups

Before we can calculate the character tables of the small Mathieu groups we need to find out their conjugacy classes, note that we are not allowing ourselves the use of a computer. This chapter is completely self contained and all work here is original. There are two main approaches detailed in this chapter.

Firstly, we assume that G is group that acts sharply 5-transitively on 12 points and see how much information about G we can obtain; this approach is detailed in Section 7.1. The author believes that it may be possible to fully describe the conjugacy class structure of such a group G, without using the fact that M_{12} is the only sharply 5-transitive group.

The second approach involves calculating the conjugacy classes of various point stabilisers and subgroups of M_{12}; as well as using information proved in Section 7.1, to obtain the a full conjugacy class description of the small Mathieu groups. This approach is detailed in sections 7.2 to 7.4.

7.1 The Conjugacy Classes of a Sharply 5-Transitive Group Acting on 12 Points

Lemma 7.1. *Let $G < Alt(12)$ be a sharply 5-transitive group on Ω, a set of 12 points, and let P be a Sylow 3-subgroup of G. Then $P \cong \mathrm{He}(3)$.*

Recall the definition of He(3) as the set of unitriangular matrices with entries in $\mathbb{F}_3$. Note that all non-trivial elements of He(3) have order 3, hence there are no elements of order 9 in G.

Proof. Observe that $|P| = 27$. There are five groups of order 27, three abelian and two non-abelian.

Let us suppose first that P is abelian. Suppose that Λ is an orbit of P in its action on Ω, and let $\lambda \in \Lambda$. Then, since P is abelian, any elements that fixes λ must fix every element in Λ. Since no element of G fixes more than 4 points, this means that $|\Lambda| = 1$ or 3. The group P must not fix more than 4 points, thus there are at least two orbits of P of size 3, call these Λ_1 and Λ_2. Now, the Orbit Stabiliser Theorem asserts that P has a subgroup P_1, of order 9, that fixes every element of Λ_1; similarly P has a subgroup P_2, of order 9, that fixes every element of Λ_2. But now $P_1 \cap P_2$ is non-trivial (by order considerations) and an element in the intersection fixes at least the 6 points of $\Lambda_1 \cup \Lambda_2$. This is a contradiction.

Suppose, then, that P is the non-abelian group of order 27 that is not He(3). This is the extraspecial group of exponent 9; it has center, Z, of order 3; it has a normal elementary-abelian subgroup, P_0, of order 9; and all the elements in $P \setminus P_0$ are of order 9. It is easy to check that the cube of each of these elements of order 9 lie in Z. One concludes immediately that any subgroup of P of order 9 must contain the center. But now observe that a point-stabiliser, G_α, in G has order divisible by 9.

Since no element of P can fix more than 4 points, a simple counting argument implies that the eight elements of P of order 3 each fix exactly 3 points, while the 18 elements of P of order 9 are all fixed-point-free. In addition the elements of Ω are partitioned into four sets of size 3, each of which is the fixed set for one of the four subgroups of P of order 3. But this contradicts the fact that a point-stabiliser of G has order divisible by 9 and we are done. □

Proposition 7.2. *Let $G < Alt(12)$ be a sharply 5-transitive group on 12 points. Then the only possible cycle types of the conjugacy classes of G are 1^{12}, $1^4 2^4$, 2^6, $1^3 3^3$, 3^4, $1^4 4^2$, $2^2 4^2$, 5^2, 6^2, $2^1 3^1 6^1$, $2^1 8^1$, $4^1 8^1$, $2^1 10^1$ and $1^1 11^1$.*

Proof. Note the following two facts about G: Firstly, $|G| = 12 \times 11 \times 10 \times 9 \times 8 = 2^6 \times 3^3 \times 5 \times 11 = 95040$. Secondly, no element of G can fix more than four points, otherwise, the 5 point stabiliser would be non-trivial. By this second fact, we can immediately ignore every cycle type that fixes more than four points.

We can immediately rule out any elements with order divisible by 7 as 7 does not divide the order of the group. No element of cycle type 12^1 is in G because an element of that cycle type would be an odd permutation and therefore not in $Alt(12)$. Any other conjugacy class must be of cycle type $1^k 2^a 3^b 4^c 5^d 6^e 8^f 9^g 10^h 11^i$.

An 11 cycle is even and fixes 1 point, therefore, $1^1 11^1$ is a possible conjugacy class of G. An element with cycle type $1^2 10^1$ is odd and cannot be in G, however, $2^1 10^1$ is even and does not fix any points, therefore $2^1 10^1$ is a possible conjugacy class of G.

By Lemma 7.1 there are no elements of order 9 or 27 in G, therefore there cannot be elements of cycle type $1^3 9^1$, $1^1 2^1 9^1$ or $3^1 9^1$ in G.

For cycles of type $1^k 2^a 3^b 4^c 8^f$ with $f \geq 1$, the cycles of the form $1^4 8^1$, $2^2 8^1$ and $1^1 3^1 8^1$ are all odd. However, $1^2 2^1 8^1$ and $4^1 8^1$ are both possible conjugacy classes of G.

For cycles of type $1^k 2^a 3^b 4^c 5^d 6^e$ it is easy to see that 6^2 and $2^1 3^1 6^1$ are possible conjugacy classes of G. Observe that $1^1 5^1 6^1$, $2^1 4^1 6^1$, $3^2 6^1$, $1^3 3^1 6^1$ and $1^4 2^2 6^1$ are odd. Squaring either of $1^4 2^2 6^1$ or $2^3 6^1$ will give a permutation that fixes too many points. So will raising $4^1 6^1$ to the 4th power. Any other permutations of this cycle type fix too many points.

For cycles of type $1^k 2^a 3^b 4^c 5^d$ with $d \geq 1$ we can see that cycles of the form $1^5 2^1 5^1$, $1^1 2^3 5^1$, $1^2 2^1 3^1 5^1$, $1^1 4^1 5^1$ and $3^1 4^1 5^1$ are odd. Consider an element of the form $1^1 2^2 5^1$. Squaring such an element would give an element of cycle type $1^7 5^1$, which fixes too many points to be in G, therefore, elements of the form $1^1 2^2 5^1$ are not in G. Elements of the form $1^4 3^1 5^1$ and $1^1 3^2 5^1$ cannot be in G as cubing them gives an element of the form $1^7 5^1$. It is easy to see that $1^2 5^2$ is a possible conjugacy class of G and that any other cycle type fixes too many points.

For cycles of type $1^k 2^a 3^b 4^c$ with $c \geq 1$, the possible conjugacy classes in G are $1^4 4^2$ and $2^2 4^2$. We can see that 4^3, $2^4 4^1$, $1^2 3^2 4^1$, $1^2 2^1 4^2$ and $1^1 2^2 3^1 4^1$ are not possible because they are odd. Moreover, raising any of $1^1 3^1 4^2$, $1^3 2^1 3^1 4^1$, $1^3 2^1 3^2 4^1$ to the fourth power will yield a permutation which fixes too many points; hence, none of these are in G. Finally, squaring $1^2 2^3 4^1$ yields a permutation which fixes too many points, so this cannot be in G either. Any other permutations of this cycle type will fix too many points.

For elements with cycle type $1^k 2^a 3^b$, squaring an element of this form will yield an element of cycle type $1^{k_2} 3^b$; whereas cubing such an element will yield an element of cycle type $1^{k_3} 2^a$. It follows that either $a = 4, 6$ and $b = 0$ or $b = 3, 4$ and $a = 0$, any other cycle type will fix too many points.s. □

Proposition 7.3. *There is at least one conjugacy class of cycle type* $1^2 5^2$ *and at least one conjugacy class of cycle type* $1^1 11^1$.

Proof. As these are the only possible cycle types containing the primes 5 and 11 and we know that G must have subgroups of size 5 and 11 it follows there must be at least one of each conjugacy class. □

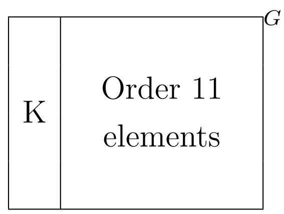
Proposition 7.4. *Let* $G < Alt(12)$ *be a sharply 5-transitive group on 12 points, then* G *must contain elements of at least one of the following cycle types:* $1^3 3^3$ *and* 3^4.

Proof. As 3^3 divides the order of G so there must be at least one conjugacy class of order 3 elements. Proposition 7.2 states that the only elements whose orders are a power of 3 are the elements of cycle type $1^3 3^3$ and 3^4. □

Lemma 7.5. *Let $G < Alt(12)$ be a sharply 5-transitive group on 12 points, let $g \in G$ such that g has cycle type $1^1 11^1$ and let $H = \langle g \rangle$, then $C_G(H) = H$ and $|N_G(H)| = 55$.*

Proof. It is immediate that $H \cong C_{11}$ and that $\mathrm{Aut}(H) \cong C_{10}$. It is easy to see that the only elements in $Alt(12)$ that commute with g are the ones generated by g, if follows that $C_G(H) = H$. By Theorem 2.16 we know that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of C_{10}, the subgroups of C_{10} are $\{1\}$, C_2, C_5 and C_{10}. Looking at Proposition 7.2 we see that the only possible cycle type that could act non-trivially on H in $\mathfrak{Grp}$ is $1^2 5^2$. The involutions and the class $2^1 10^1$ move 8 or 12 points, which will not preserve the structure of H.

Suppose $N_G(H) = H$, then the number of Sylow 11-subgroups n_{11} would equal $|G|/|H| = |G|/11$. As all of the Sylow 11-subgroups intersect trivially the number of elements of order 11 is equal to $\frac{10}{11}|G|$. Let K be a point stabiliser of G acting on 12 points, then $|K| = \frac{1}{11}|G|$. It follows that 11 does not divide $|K|$ and K has no order 11 elements.



But then K has no conjugates as there is nothing to conjugate it to. Hence K must be normal in G. Now, every point stabiliser must be conjugate to K, but this means that K is the stabiliser of every point, a contradiction. From this we can deduce that $N_G(H)/C_G(H) \cong C_5$ and it follows that $|N_G(H)| = |C_{11}| \times |C_5| = 55$. □

Proposition 7.6. *Let $G < Alt(12)$ be a sharply 5-transitive group on 12 points, then there are exactly 2 conjugacy classes of elements with cycle type $1^1 11^1$.*

Proof. Let $g \in G$ be an element of order dividing 11; then g must be an element of cycle type $1^1 11^1$ because G is a permutation on 12 points. Now $C_G(g) = \langle g \rangle \cong C_{11}$, hence, by Theorem 2.9 we can see that $|g^G| = 12 \times 10 \times 9 \times 8$ when acting by conjugation. Now, let $P \in Syl_{11}(G)$ and let n_{11} be the number of Sylow 11-subgroups. As the multiplicity of 11 in $|G|$ is 1, the conjugates of P intersect trivially, it follows that the number of elements of order 11 is equal to $10 \times n_{11}$. By Lemma 7.5 we see that $|N_G(\langle g \rangle)| = 55$, hence, Theorem 2.16 gives $n_{11} = \frac{12 \times 11 \times 10 \times 9 \times 8}{11 \times 5} = 12 \times 9 \times 8 \times 2$. Moreover, the number of elements of order 11 is equal to $10 n_{11} = 10 \times (12 \times 9 \times 8 \times 2) = 2 \times |g^G|$. □

Lemma 7.7. *Let $G < Alt(12)$ be a sharply 5-transitive group on 12 points, then G has exactly one conjugacy class of elements of cycle type $1^2 5^2$. Moreover, for g of cycle type $1^2 5^2$ we have $C_G(g) \cong C_{10}$ and $|N_G(\langle g \rangle)| = 40$.*

Proof. Let g be an element of cycle type $1^2 5^2$ and let $H = \langle g \rangle$. As g is composed of two 5 cycles, it is easy to see that $\langle g \rangle \cong C_5$. Now, G is 5-transitive, so we can pick an element $h \in G$, such that when h conjugates g, h sends one of the 5 cycles of g to its inverse. Let g^h denote this conjugate, if $g^h g \neq 1$ then $g^h g$ fixes 5 points, a contradiction. Therefore $g^h g = 1$ and $g^h = g^{-1}$. The same argument can be used to show that g is conjugate to g^2 and g^3. It follows that there is a unique conjugacy class of elements of cycle type $1^2 5^2$.

The elements that conjugate g to its powers must normalise H, therefore $|N_G(H)| \geq 4|H|$. As $N_G(H)/C_G(H)$ embeds into $\mathrm{Aut}(H)$ and we have described a full set of automorphisms for H, it follows that $N_G(H)/C_G(H) \cong \mathrm{Aut}(H)$. Moreover, $|C_G(H)| = |H| = 5$ and $|N_G(H)| = 20$, therefore the number of Sylow-5 subgroups n_5 must equal 4752 which does not equal 1 (mod 5). Hence $|N_G(H)| > 4|H|$ and $|C_G(H)| > |H|$.

An element k where $k^2 = g$ of cycle type $2^1 10^1$ could centralise H and the subgroup generated by k would be the maximal possible centraliser of H in G. In fact this is the only possible centraliser for H in G as other elements in $C_{Alt(12)}(H)$ fix too many points. We can conclude that $C_G(H) \cong C_{10}$ and $|N_G(H)| = 40$.

Furthermore, let n_5 be the number of Sylow 5-subgroups and note that they intersect trivially. We have $|N_G(H)| = 40$ hence $n_5 = 2376$ by Theorem 2.16. The number of elements of cycle type $1^2 5^2$ in G is $4n_5 = 9504$. □

Corollary 7.8. *Let $G < Alt(12)$ be a sharply 5-transitive group on 12 points, then G contains exactly one conjugacy class of cycle type $2^1 10^1$ and at least one conjugacy class of cycle type 2^6.*

Proof. The subgroup $K = \langle k \rangle \cong C_{10}$ must contain an order 10 element, the only possible cycle type that is order 10 is $2^1 10^1$. Raising such an element to the fifth power gives an element of cycle type 2^6. It is easy to see that $C_G(K) = K$ and $|N_G(H)| = |N_G(K)|$. There are 4 elements of order 10 in K given by $\{k, k^3, k^7, k^9\}$, it follows that the number of elements of order 10 is equal to $4n_5 = 9504$. By Theorem 2.9, under the conjugation action, $|k^G| = 9504$, hence, every element of cycle type $2^1 10^1$ in G is conjugate. □

We shall summarise the key results from this section. If $G < Alt(12)$ is a sharply 5-transitive group on 12 points, then G contains exactly one class of elements of the following cycles types: 1^{12}, $1^2 5^2$ and $2^1 10^1$. Moreover, both of the later classes have 9504 elements. G contains two conjugacy classes of cycle type $1^1 11^1$, each with 8640 elements. G also contains at least one conjugacy class of cycle type 2^6. Finally, the Sylow 3-subgroup of G must be He(3) and G must contain at least one of the conjugacy classes $1^3 3^3$ or 3^4.

7.2 The Conjugacy Classes of M_9 and M_{10}

Cycle Type	1^9	1^12^4	3^3	1^14^2	1^14^2	1^14^2
Size	1	9	8	18	18	18
Centraliser	72	8	9	4	4	4

Table 7.1: The conjugacy classes of M_9.

Theorem 7.9. *The conjugacy classes of M_9 are exactly those given in Table 7.1.*

Proof. Recall that $M_9 \cong \mathbb{F}_3^2 \rtimes_\eta Q_8 = \langle \phi_1, \phi_2, \theta, \lambda \rangle$ with $\mathbb{F}_3^2 \lhd M_9$. It follows that there can only be elements of orders containing powers of 2 or 3. However, there are not any elements of order 6 in M_9. To show this, let $g \in M_9$ be an order 6 element. Such an element must be the product of an involution with an order 3 element; that is $\langle g \rangle \cong C_6 \cong C_3 \times C_2$. But observe, an involution in M_9 corresponds to $-I_2$, where I_2 is the identity matrix. Hence, an involution does not fix any non zero vector, therefore, any product of the elements would be a non trivial semi-direct product, which does not contain elements of order 6. It follows that every element is of order 1, 2, 3 or 4.

Under the action of Q_8, the order 3 elements form one single conjugacy class of M_9 with cycle type 3^3. There are 8 such elements and their centraliser is isomorphic to the group C_3^2. Any remaining classes must come from the copies of Q_8 in the group. As Q_8 is the Sylow 2-subgroup of M_9, it is easy to deduce that there are 9 copies of Q_8, all of which are conjugate.

Assume that two or more of the conjugates of Q_8 do not intersect trivially. It follows that this intersection must contain an involution. Moreover, this involution must be the only involution in each Q_8. But now the centraliser of the involution is a power of 2 bigger than 8; this does not divide $|M_9|$, a contradiction. Hence, every copy of Q_8 intersects trivially.

We can now obtain the numbers of the order 2 and 4 elements. There are 9 elements of type 1^12^4, all of which are conjugate. There are $3 \times 18 = 54$ elements of type 1^12^4, each one of these is centralised by a subgroup of size 4. An application of Theorem 2.9 and a simple counting argument shows that there must be 3 classes of elements 1^12^4. □

Cycle Type	1^{10}	1^22^4	$1^13^3_A$	$1^13^3_B$	1^24^2	5^2_A	5^2_B
Size	1	45	40	40	90	72	72
Centraliser	360	8	9	9	4	5	5

Table 7.2: The conjugacy classes of $Alt(6)$ acting on 10 points.

Theorem 7.10. *The conjugacy classes of $Alt(6)$ acting on 10 points are exactly those given in Table 7.2.*

Proof. We know that $Alt(6)$ contains every class of even permutations. The splitting criteria (Theorem 2.17) states that the class of elements of order 5 will split. Now, the result immediately follows from the construction of the action of $Alt(6)$ given in Section 6.2. □

Lemma 7.11. *Let $P \in Syl_2(Alt(6))$ then $N_{Alt(6)}(P) = P$.*

Proof. Let n_2 be the number of Sylow 2-subgroups of $Alt(6)$. By Theorem 6.8 we have $P \cong D_8$ and $|P| = 8$. By Theorem 2.21 we have $n_2 \equiv 1 \mod 2$ and that n_2 divides 45. There are 6 possibilities for n_2: 1, 3, 5, 9, 15 or 45. $Alt(6)$ is simple so $n_2 \neq 1$. Moreover, $Alt(6)$ does not have subgroups of size 40, 72 or 120, therefore $n_2 \neq 9$, $n_2 \neq 5$ and $n_2 \neq 3$. The only subgroup of size 24 is $Sym(4)$, but D_8 is not normal in $Sym(4)$. Therefore $n_2 = 45$ and P is self normalising. □

Cycle Type	1^{10}	$1^2 2^4$	$1^1 3^3$	$1^2 4^2_A$	$1^2 4^2_B$	5^2	$2^1 8^1_A$	$2^1 8^1_B$
Size	1	45	80	90	180	144	90	90
Centraliser	720	16	9	8	4	5	8	8

Table 7.3: The conjugacy classes of M_{10}.

Theorem 7.12. *The conjugacy classes of M_{10} are exactly those given in Table 7.3.*

Proof. Recall that $M_{10} = \langle \phi_1, \phi_2, \theta, \psi, \lambda \rangle$. A consequence of Theorem 6.8 is there are no order 2 elements in $\lambda Alt(6)$. Hence, there are exactly 45 elements of order 2 in one conjugacy class $1^2 2^4$ (these elements are from $Alt(6)$).

By order considerations every order 3 element must be contained inside a copy of M_9. As M_9 is a one point stabiliser of M_{10} they are all conjugate. Moreover, these elements move 9 points; therefore each element is contained in exactly one copy of M_9. It follows that there is only one class of elements 3^3 with 80 elements.

Recall that $M_{10} = Alt(6) \cup \lambda Alt(6)$ and that elements of $\lambda Alt(6)$ conjugate elements of $Alt(6)$, to elements of $Alt(6)$. Let g be an element of order 5 in $Alt(6)$, then $|C_{Alt(6)}(g)| = |C_{M_{10}}(g)| = 5$. By Theorem 2.9 it follows that the two classes of order 5 elements in $Alt(6)$ are fused into one class of size 144. It is also immediate that there is a class of elements (from $Alt(6)$) of cycle type $1^2 4^2$ containing 90 elements (class $1^2 4^2_A$).

By Lemma 7.11 a Sylow 2-subgroup of $Alt(6)$ is self normalising, has size 8 and index 45. Now, the Sylow 2-subgroups of M_{10} are non split extensions of the Sylow 2-subgroups of $Alt(6)$. It immediately follows that the Sylow 2-subgroups of M_{10} are of size 16, have index 45 and are self normalising.

Let $Q \in Syl_2(M_{10})$ and $P < Q$ with $P \in Syl_2(Alt(6))$, so $Q = P \cup \lambda P$. By Theorem 6.8 there are 4 elements of cycle type $1^2 4^2$ and 4 elements of cycle type $2^1 8^1$ in λP. Moreover, in Q there are 2 classes of cycle type $1^2 4^2$, one with 2 elements and one with 4. There are also 2 classes of elements of cycle type $2^1 8^1$. As all Sylow subgroups are conjugate, we can deduce that there is exactly two classes of elements with cycle type $2^1 8^1$, each with 90 elements. Finally, we can see a second class of elements of cycle type $1^2 4^2$, containing 180 elements (class $1^2 4^2_B$). □

7.3 The Conjugacy Classes of M_{11}

Cycle Type	1^{11}	$1^3 2^4$	$1^2 3^3$	$1^3 4^2$	$1^1 5^2$	$2^1 3^1 6^1$	$1^1 2^1 8^1_A$	$1^1 2^1 8^1_B$	11^1_A	11^1_B
Size	1	165	440	990	1584	1320	990	990	720	720
Centraliser	7920	24	18	8	5	6	8	8	11	11

Table 7.4: The conjugacy classes of M_{11}.

Theorem 7.13. *The conjugacy classes of M_{11} are exactly those in Table 7.4.*

Proof. Recall that M_{11} acts sharply 4 transitively on a set of size 11, and the 3 point stabiliser of the action is isomorphic to Q_8. Hence there are 165 conjugate copies of Q_8 in M_{11}. We immediately see that there are 165 elements of cycle type $1^2 2^4$, and that they are all conjugate. Moreover, we see that there are $165 \times 6 = 990$ elements of cycle type $1^3 2^4$. By considering the centraliser of such an element we can deduce that these are all conjugate.

We can use a similar argument for elements of cycle type $1^2 3^3$ in M_9 and for elements of cycle type $1^1 5^2$ in M_{10}. We see that there is one class of each, with 440 and 1584 elements respectively. We can also see that there are 2×990 elements of cycle type $1^1 2^1 8^1$. By considering the centraliser of such an element, we can deduce that there are exactly two equally sized classes of these elements.

An argument almost identical to the one given in Proposition 7.6 shows that there are 2 classes of cycle type 11^1, each with 720 elements. Finally, an element of cycle type $2^1 3^1 6^1$ has a centraliser of size 6, hence, there are 1320 elements of this type. □

7.4 The Conjugacy Classes of M_{12}

We begin this section by remarking that $P = \langle \theta, \lambda, \psi, \sigma, \zeta \rangle$, where $\zeta = \mu\psi\sigma\mu^{-1}$, is a Sylow 2-subgroup of M_{12}. The conjugacy classes of P are given in Table 7.5. We note that $|P| = 64$ and that $\mathrm{Z}(P) \cong C_2$.

Representative Element	Size	Centraliser
1	1	64
$(23)(49)(57)(68)$	1	64
$(01)(57)(68)(X\infty)$	2	32
$(0\infty)(1X)(49)(68)$	4	16
$(49)(58)(67)(X\infty)$	8	16
$(01)(25)(37)(48)(69)(X\infty)$	4	16
$(0\infty)(1X)(26)(38)(47)(59)$	4	16
$(0\infty)(1X)(24)(39)(56)(78)$	4	16
$(2439)(5678)$	2	32
$(2537)(4896)$	4	16
$(0\infty 1X)(5678)$	4	16
$(01)(X\infty)(2439)(5876)$	2	32
$(23)(49)(0\infty 1X)(5876)$	4	16
$(0\infty)(1X)(2537)(4698)$	4	16
$(X\infty)(25983746)$	8	8
$(0\infty 1X)(25483796)$	8	8

Table 7.5: The conjugacy classes of a Sylow 2-subgroup of M_{12}.

Cycle Type	Centraliser	Class Size	Cycle Type	Centraliser	Class Size
1^{12}	95040	1	$1^2 5^2$	10	9504
$1^4 2^4$	192	495	6^2	12	7920
2^6	240	396	$1^1 2^1 3^1 6^1$	6	15840
$1^3 3^3$	54	1760	$1^2 2^1 8^1$	8	11880
3^4	36	2640	$4^1 8^1$	8	11880
$1^4 4^2$	32	2970	$2^1 10^1$	10	9504
$2^2 4^2$	32	2970	$1^1 11^1$	11	8640×2

Table 7.6: The conjugacy classes of M_{12}.

Theorem 7.14. *The conjugacy classes of M_{12} are exactly those given in Table 7.6*

Proof. From Section 7.1 we immediately obtain counts and centraliser orders of the classes $1^2 5^2$, $2^1 10^1$ and both classes of $1^1 11^1$. We can adapt the point stabiliser arguments from Theorem 7.13. Considering Q_8 as a 4 point stabiliser, we see there are 495 elements of cycle type $1^4 2^4$ and they are all conjugate. We also see that there are 2970 elements of type 4^2, and using the class structure of M_{11}, we see these are all conjugate.

By considering M_9 as a 3 point stabiliser, we immediately see there are 1760 elements of cycle type $1^3 3^3$ in a single conjugacy class of M_{12}. Moreover, considering M_{11} as the 1 point stabiliser immediately gives 15840 elements of cycle type $1^1 2^1 3^1 6^1$ in a single conjugacy class of M_{12}.

By considering M_{10}, we obtain that there are 11880 elements of cycle type $1^2 2^1 8^1$ in M_{12}. We now need to use information about the Sylow 2-subgroups of M_{12}. There is exactly one class of elements $1^2 2^1 8^1$ and one class of elements $4^1 8^1$ in any Sylow 2-subgroup; hence, there is exactly one of each in M_{12}, each with 11880 elements. By considering the number of elements of type $1^4 4^2$, we see that there are 2970 elements of type $2^2 4^2$.

Let $P_3 = \langle \phi_1, \phi_2, \sigma\mu \rangle$, then $P_3 \in Syl_3(M_{12})$. Note that $Z(P_3) = \langle \phi_2 \rangle$, hence, $Z(P_3)$ only contains elements of cycle type $1^3 3^3$. Now, observe that the point stabiliser of $X, 0$ and ∞ in P_3, it is exactly $\langle \phi_1, \phi_2 \rangle$.

We note at this point that we have described 84084 elements of M_{12}, there are 10956 elements left. Now, let g be an element of cycle type 6^2, then $|C_{Alt(12)}(g)| = \frac{6\times 6\times 2}{2} = 36$. If we remove any elements which fix more than 5 points from $C_{Alt(12)}(g)$, we are left with 31 elements, 4 of which are order 3. Now, $|C_{M_{12}}(g)|$ must divide $|C_{Alt(12)}(g)|$, therefore $|C_{M_{12}}(g)| = 3, 6, 12$ or 18. If $|C_{M_{12}}(g)| = 18$ then there must be 8 elements of order 3, but there are at most 4 such elements. Hence, $|C_{M_{12}}(g)| = 3, 6$ or 12, by considering $|M_{12}|$, we see that $|C_{M_{12}}(g)| = 12$, otherwise there will be too many elements in M_{12}. We verify the existence of such an element by considering that $\theta\phi_2\psi\mu\sigma = (129\infty X0)(357864)$.

There are 3036 elements left. Let g be an element of cycle type 3^4; such an element can be obtained by squaring an element of cycle type 6^2. Now, $|C_{Alt(12)}(g)| = 972$ and as $\mathrm{Z}(P_3)$ only contains elements of cycle type $1^3 3^3$, we have $|C_P(g)| = 9$. Moreover, $|C_{M_{12}}|$ is divisible by 9 but not 27. It follows that $|C_{M_{12}}(g)| = 9, 18$ or 36. By order considerations, $|C_{M_{12}}(g)| = 36$ because the other options give too many elements. Hence, there is one class of cycle type 3^4 containing 2640 elements.

There are 396 elements left. Let g be an element of cycle type 2^6. By considering the centraliser of g in a Sylow 2-subgroup of M_{12}, we immediately see that $|C_{M_{12}}(g)|$ is not divisible by 32. Moreover, $|C_{M_{12}}(g)|$ is not divisible by 11, but is divisible by 3. Hence, $|C_{M_{12}}(g)| \geq 240$. Now, consider the Sylow 3-subgroup of $C_{Alt(12)}(g)$, it is isomorphic to C_3^2, but contains elements that fix more than 5 points. It follows that $|C_{M_{12}}(g)| = 240$. $\square$

Chapter 8

The Character Tables of the Small Mathieu Groups

In this chapter we will construct the character tables of M_9, M_{10}, M_{11} and M_{12}. An inspection of the character tables of M_{11} and M_{12} yields the following theorem.

Theorem 8.1. *The groups M_{11} and M_{12} are simple.*

Proof. By inspection of Table 8.3 we see that every character has trivial kernel. By Theorem 4.22 the only normal subgroups of M_{11} are M_{11} and $\{1\}$. The same observation can be seen from Table 8.4. Hence, the only normal subgroups of M_{12} are M_{12} and $\{1\}$. □

8.1 The Character Table of M_9 and M_{10}

We shall begin with M_9. By Theorem 7.9 we know M_9 has 6 conjugacy classes and hence 6 irreducible characters. As $C_3 \times C_3$ is normal in M_9 and $M_9/(C_3 \times C_3) \cong Q_8$, we can lift 3 linear characters and a 2 dimensional character from Q_8. The final character χ_5 is given by the 2-transitive action of M_9 on 9 points (Theorem 4.37). The character table is given in Table 8.1.

M_9	1^9	2^4	3^3	4_A^2	4_B^2	4_C^2
χ_0	1	1	1	1	1	1
χ_1	1	1	1	-1	1	-1
χ_2	1	1	1	1	-1	-1
χ_3	1	1	1	-1	-1	1
χ_4	2	-2	2	0	0	0
χ_5	8	0	-1	0	0	0

Table 8.1: The character table of M_9.

Next, we will calculate the character table of M_{10}. First note that M_{10} has 8 conjugacy classes, hence 8 irreducible characters. Recall that $Alt(6) \triangleleft M_{10}$. So we can lift a character from $M_{10}/Alt(6) \cong C_2$.

M_{10}	1^{10}	2^4	3^3	4_A^2	4_B^2	5^2	$2^1 8_A^1$	$2^1 8_B^1$
χ_0	1	1	1	1	1	1	1	1
χ_1	1	1	1	1	-1	1	-1	-1

We also have the permutation character χ_2, given by the 3-transitive action of M_{10} on 10 points. Moreover, let $\chi_3 = \chi_1\chi_2$, then χ_3 is also irreducible.

χ_2	9	1	0	1	1	-1	-1	-1
χ_3	9	1	0	1	-1	-1	1	1

We will try inducing characters from $Alt(6)$. Let χ be a character of $Alt(6)$; then using the centraliser orders and Theorem 5.13 we have:

$$(\chi \uparrow M_{10})(g) = \begin{cases} 2\chi(g) & \text{if } g \in 1^{10}, 2^4, 4^2; \\ \chi(g_A) + \chi(g_B) & \text{if } g \in 3^3, 5^2; \\ 0 & \text{otherwise.} \end{cases}$$

Let χ_4 be the induced character of a 5 dimensional character of $Alt(6)$ and let χ_7 be the induced character of an 8 dimensional character. We obtain the following:

χ_4	10	2	1	-2	0	0	0	0
χ_7	16	0	-2	0	0	1	0	0

As $\langle \chi_4, \chi_4 \rangle = 1$ and $\langle \chi_7, \chi_7 \rangle = 1$, we see that χ_4 and χ_7 are irreducible. As an element of cycle type $2^1 8^1$ is not conjugate to its inverse, by Theorem 4.34 the remaining two characters are complex conjugates with dimension d. Subtracting the squares of the known dimensions from $|M_{10}| = 720$, gives 200. Hence, $2d^2 = 200$, and $d = 10$. These two 10 dimensional characters can now be easily calculated using the Schur Orthogonality Relations. Table 8.2 gives the full character table of M_{10}.

M_{10}	1^{10}	2^4	3^3	4^2_A	4^2_B	5^2	$2^1 8^1_A$	$2^1 8^1_B$
χ_0	1	1	1	1	1	1	1	1
χ_1	1	1	1	1	-1	1	-1	-1
χ_2	9	1	0	1	1	-1	-1	-1
χ_3	9	1	0	1	-1	-1	1	1
χ_4	10	2	1	-2	0	0	0	0
χ_5	10	-2	1	0	0	0	ω	$\overline{\omega}$
χ_6	10	-2	1	0	0	0	$\overline{\omega}$	ω
χ_7	16	0	-2	0	0	1	0	0

Table 8.2: The character table of M_{10}, where $\omega = \sqrt{-2}$.

8.2 The Character Table of M_{11}

8.2.1 The Permutation Character and Tensor Products

From the construction we see that M_{11} acts 4-transitively on a set of size 11. By considering the number of fixed points of each conjugacy class we obtain a 10 dimensional irreducible character χ_1 (Theorem 4.37).

M_{11}	1^{11}	2^4	3^3	4^2	5^2	$2^1 3^1 6^1$	$2^1 8^1_A$	$2^1 8^1_B$	11^1_A	11^1_B
χ_0	1	1	1	1	1	1	1	1	1	1
χ_1	10	2	1	2	0	-1	0	0	-1	-1

Now, let χ_S and χ_A be the symmetric and antisymmetric decomposition of χ_1^2.

χ_S	55	7	1	3	0	1	1	1	0	0
χ_A	45	-3	0	1	0	0	-1	-1	1	1

A quick calculation gives $\langle \chi_S, \chi_S \rangle = 3$ and $\langle \chi_A, \chi_A \rangle = 1$. Moreover, $\langle \chi_S, \chi_0 \rangle = 1$ and $\langle \chi_S, \chi_1 \rangle = 1$. Define $\chi_8 = \chi_A$ and $\chi_7 = \chi_S - \chi_0 - \chi_1$ and note that $\langle \chi_7, \chi_7 \rangle = 1$. Hence we have found 2 new irreducible characters of M_{11}.

χ_7	44	4	-1	0	-1	1	0	0	0	0
χ_8	45	-3	0	1	0	0	-1	-1	1	1

8.2.2 Induction from $Alt(6)$

We will now try inducing characters from the subgroup $Alt(6)$. Let χ be a character of $Alt(6)$, then using the centraliser orders and Theorem 5.13 we have:

$$(\chi \uparrow M_{11})(g) = \begin{cases} 22\chi(g) & \text{if } g \in 1^{11}; \\ 6\chi(g) & \text{if } g \in 2^4; \\ 2\chi(g) & \text{if } g \in 4^2; \\ \chi(g_A) + \chi(g_B) & \text{if } g \in 3^3, 5^2; \\ 0 & \text{otherwise.} \end{cases}$$

Let χ_X be the induced character of the trivial character of $Alt(6)$ and let χ_Y be the induced character of a 5 dimensional character.

χ_X	22	6	4	2	2	0	0	0	0	0
χ_Y	110	6	2	-2	0	0	0	0	0	0

We see that $\langle \chi_X, \chi_X \rangle = 3$ and $\langle \chi_Y, \chi_Y \rangle = 3$. After taking the inner product of χ_X with each of the known irreducible characters we find that $\langle \chi_X, \chi_0 \rangle = 1$ and $\langle \chi_X, \chi_1 \rangle = 1$. Let $\chi_4 = \chi_X - \chi_0 - \chi_1$, then $\langle \chi_4, \chi_4 \rangle = 1$; hence χ_4 is a new irreducible character. Repeating the process with χ_Y we obtain that $\langle \chi_Y, \chi_4 \rangle = 1$ and $\langle \chi_Y, \chi_7 \rangle = 1$. Now, let $\chi_9 = \chi_Y - \chi_4 - \chi_7$ then $\langle \chi_9, \chi_9 \rangle = 1$ and we have a new irreducible character.

χ_4	11	3	2	-1	1	0	-1	-1	0	0
χ_9	55	-1	1	-1	0	-1	1	1	0	0

8.2.3 Schur Orthogonality

The remaining four characters come in complex conjugate pairs. We can deduce this by using the fact that elements of cycle type $2^1 8^1$ and 11^1 are not conjugate to their inverses and applying Theorem 4.34. Given that each pair will have the same dimension, we can attempt to calculate the dimension of these 4 remaining characters. Let the dimension of the first pair be d_1 and the dimension of the second pair be d_2.

We have $d_1^2 + d_1^2 + d_2^2 + d_2^2 = 712$, hence $d_1^2 + d_2^2 = 356$. By an exhaustive search we find that 356 can be expressed as the sum of two squares in exactly one way, that is $356 = 10^2 + 16^2$. It immediately follows that $d_1 = 10$ and $d_2 = 16$.

Let $\chi_2(1) = \chi_3(1) = 10$ and $\chi_5(1) = \chi_6(1) = 16$. We will now use the column relations to calculate the character values for the remaining conjugacy classes. We let the characters take the following values:

χ_2	11	x_1	x_2	x_3	x_4	x_5	x_6	$\overline{x_6}$	x_7	$\overline{x_7}$
χ_3	11	x_1	x_2	x_3	x_4	x_5	$\overline{x_6}$	x_6	$\overline{x_7}$	x_7
χ_5	16	y_1	y_2	y_3	y_4	y_5	y_6	$\overline{y_6}$	y_7	$\overline{y_7}$
χ_6	16	y_1	y_2	y_3	y_4	y_5	$\overline{y_6}$	y_6	$\overline{y_7}$	y_7

Substituting the column containing x_i for $i = 1, \ldots 5$ and the first column of the character table into (4.4) we can obtain values for x_i and y_i. We shall demonstrate this with x_1.

$$1 + 20 + 2 \times 10x_1 + 33 + 2 \times 16y_1 + 176 - 135 - 55 = 0$$

$$5x_1 + 8y_1 = -10 \tag{8.1}$$

Substituting the column containing x_1 into (4.4) twice gives:

$$1 + 4 + 2x_1^2 + 9 + 2y_1^2 + 16 + 9 + 1 = 48$$

$$x_1^2 + y_1^2 = 4 \tag{8.2}$$

Solving (8.1) and (8.2), we obtain two solutions $x_1 = -2,\ y_1 = 0$ and $x_1 = \frac{78}{89},\ y_1 = \frac{-160}{89}$. The second set of these cannot be expressed as sum of 2nd roots of unity. Hence, $x_1 = -2$ and $y_1 = 0$. Continuing in this manner we obtain:

χ_2	10	-2	1	0	0	1	x_6	$\overline{x_6}$	x_7	$\overline{x_7}$
χ_5	16	0	-2	0	1	0	y_6	$\overline{y_6}$	y_7	$\overline{y_7}$

The remaining values $x_6,\ x_7,\ y_6$, and y_7 can be calculated by repeat applications of the row and column relations. We find that $x_6 = \sqrt{-2}$, $x_7 = -1$, $y_6 = 0$ and $y_7 = \frac{1}{2}(-1 + \sqrt{-11})$. Table 8.3 shows the complete character table.

M_{11}	1^{11}	2^4	3^3	4^2	5^2	$2^1 3^1 6^1$	$2^1 8^1_A$	$2^1 8^1_B$	11^1_A	11^1_B
χ_0	1	1	1	1	1	1	1	1	1	1
χ_1	10	2	1	2	0	-1	0	0	-1	-1
χ_2	10	-2	1	0	0	1	α	$\overline{\alpha}$	-1	-1
χ_3	10	-2	1	0	0	1	$\overline{\alpha}$	α	-1	-1
χ_4	11	3	2	-1	1	0	-1	-1	0	0
χ_5	16	0	-2	0	1	0	0	0	β	$\overline{\beta}$
χ_6	16	0	-2	0	1	0	0	0	$\overline{\beta}$	β
χ_7	44	4	-1	0	-1	1	0	0	0	0
χ_8	45	-3	0	1	0	0	-1	-1	1	1
χ_9	55	-1	1	-1	0	-1	1	1	0	0

Table 8.3: The character table of M_{11}, where $\alpha = \sqrt{-2}$ and $\beta = \frac{1}{2}(-1 + \sqrt{-11})$.

8.3 The Character Table of M_{12}

We begin by noting that M_{12} has 15 conjugacy classes and 15 irreducible characters, one of which is the trivial character χ_0.

8.3.1 The Permutation Character and Tensor Products

By construction M_{12} acts 5-transitively on a set of size 12. Hence, by Theorem 4.37 get the permutation character χ_1.

M_{12}	1^{12}	2^4	2^6	3^3	3^4	4^2	$2^2 4^2$	5^2	$2^1 3^1 6^1$	6^2	$2^1 8^1$	$4^1 8^1$	$2^1 10^1$	11^1_A	11^1_B
χ_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_1	11	−1	3	2	−1	3	−1	1	0	−1	−1	1	−1	0	0

Now, let χ_S and χ_A be the symmetric and antisymmetric decompositin of χ_1^2.

χ_S	66	10	6	3	0	6	2	1	1	0	2	0	1	0	0
χ_A	55	−1	−5	1	1	3	−1	0	−1	1	−1	1	0	0	0

A quick calculation gives $\langle \chi_S, \chi_S \rangle = 3$ and $\langle \chi_A, \chi_A \rangle = 1$. Moreover, $\langle \chi_S, \chi_0 \rangle = 1$ and $\langle \chi_S, \chi_1 \rangle = 1$. Define $\chi_8 = \chi_A$ and $\chi_6 = \chi_S - \chi_0 - \chi_1$. We have found two new irreducible characters of M_{12}. Squaring these characters, however, is not a viable plan; the characters obtained have dimensions 2916 and 3025.

χ_6	54	6	6	0	0	2	2	−1	0	0	0	0	1	−1	−1
χ_8	55	−1	−5	1	1	3	−1	0	−1	1	−1	1	0	0	0

8.3.2 Induction from M_{11}

We will now try inducing characters from the subgroup M_{11}. Let χ be a character of M_{11}; then using the centraliser orders and Theorem 5.13 we have:

$$(\chi \uparrow M_{12})(g) = \begin{cases} 12\chi(g) & \text{if } g \in 1^{12}; \\ 4\chi(g) & \text{if } g \in 2^4, 4^2; \\ 3\chi(g) & \text{if } g \in 3^3; \\ 2\chi(g) & \text{if } g \in 5^2; \\ \chi(g) & \text{if } g \in 2^1 3^1 6^1, 11^1_A, 11^1_B; \\ \chi(g_A) + \chi(g_B) & \text{if } g \in 2^1 8^1; \\ 0 & \text{otherwise.} \end{cases}$$

Inducing the trivial character of M_{11} gives a character equal to $\chi_0 + \chi_1$. Inducing the integer valued 10-dimensional character gives a character equal to $\chi_1 + \chi_6 + \chi_8$. Now, let χ_{12} be the induced character of a complex valued 10-dimensional character of M_{11}. We see that $\langle \chi_{12}, \chi_{12} \rangle = 1$, therefore, χ_{12} is irreducible.

Let χ_V be the character obtained by inducing the 55-dimensional character of M_{11}. Finally, let χ_B be the induced complex valued 16-dimensional character of M_{11} and let $\omega = \frac{1}{2}(-1 + \sqrt{-11})$, then we have:

χ_{12}	120	-8	0	3	0	0	0	0	0	1	0	0	0	-1	-1
χ_V	660	-4	0	3	0	0	-4	0	0	-1	2	0	0	0	0
χ_B	192	0	0	-6	0	0	0	2	0	0	0	0	0	ω	$\overline{\omega}$

Note that $\langle \chi_B, \chi_B \rangle = 2$ and $\langle \chi_V, \chi_V \rangle = 6$, but the inner product of χ_B with any known irreducible is 0. We have $\chi_V, \chi_{12} = 1$, but the inner product is 0 for any other known irreducibles. These do not give us any new irreducible characters, but we will use these characters later.

8.3.3 Restriction from $Sym(12)$

Using the Frobenius character formula, it is possible to construct low dimensional characters of $Sym(12)$ evaluated over the conjugacy classes of M_{12}; we will not detail this because it is a time consuming and repetitive process. We note that $(\chi_{(11,1)} \downarrow M_{12}) = \chi_1$, $(\chi_{(10,2)} \downarrow M_{12}) = \chi_2$ and $(\chi_{(10,1,1)} \downarrow M_{12}) = \chi_3$. For ease of notation let $\lambda_A = (9,1,1,1)$ and $\lambda_B = (8,1,1,1,1)$. We can then construct the following characters of M_{12}.

M_{12}	1^{12}	2^4	2^6	3^3	3^4	4^2	$2^2 4^2$	5^2	$2^1 3^1 6^1$	6^2	$2^1 8^1$	$4^1 8^1$	$2^1 10^1$	11^1_A	11^1_B
$\chi_{(9,3)}$	154	10	-6	1	4	-2	-2	-1	1	0	0	0	-1	0	0
χ_{λ_A}	165	-11	5	3	3	1	1	0	1	-1	-1	-1	0	0	0
$\chi_{(8,4)}$	275	11	15	5	-4	-1	3	0	-1	0	-1	1	0	0	0
$\chi_{(7,5)}$	297	9	-15	0	0	5	-3	2	0	0	-1	-1	0	0	0
$\chi_{(3,2,1)}$	320	0	0	-4	-4	0	0	0	0	0	0	0	0	1	1
χ_{λ_B}	330	-6	10	6	-3	-2	-2	0	1	0	0	0	0	0	0

We first check the inner product of each character with itself and then with each of the known irreducibles. We find that the $\langle \chi_{\lambda_A}, \chi_{\lambda_A} \rangle = 2$ and $\langle \chi_{\lambda_A}, \chi_{12} \rangle = 1$, define $\chi_5 = \chi_{\lambda_A} - \chi_{12}$ and note that $\langle \chi_5, \chi_5 \rangle = 1$. Hence, we have found a new irreducible character of M_{12}.

χ_5	45	-3	5	0	3	1	1	0	0	-1	-1	-1	0	1	1

We shall next check χ_{λ_B}, we find that $\langle \chi_{\lambda_B}, \chi_{\lambda_B} \rangle = 3$ and that $\langle \chi_{\lambda_B}, \chi_{12} \rangle = 1$. Define $\chi_X = \chi_{\lambda_B} - \chi_{12}$. We will also look at $\chi_{(8,4)}$. We find that $\langle \chi_{(8,4)}, \chi_{(8,4)} \rangle = 4$ and that $\langle \chi_{(8,4)}, \chi_6 \rangle = 1$, define $\chi_Y = \chi_{(8,4)} - \chi_6$. Checking the inner products of χ_X and χ_Y with themselves and each other we obtain $\langle \chi_X, \chi_X \rangle = 2$, $\langle \chi_Y, \chi_Y \rangle = 3$ and $\langle \chi_X, \chi_Y \rangle = 2$. Define $\chi_2 = \chi_Y - \chi_X$, we find that $\langle \chi_2, \chi_2 \rangle = 1$; we have found a new irreducible character of M_{12}.

χ_Y	221	5	9	5	−4	−3	1	1	−1	0	−1	1	−1	1	1
χ_X	210	2	10	3	−3	−2	−2	0	1	−1	0	0	0	1	1
χ_2	11	3	−1	2	−1	−1	3	1	0	−1	−1	1	−1	0	0

Let χ_S and χ_9 be the symmetric and antisymmetric decomposition of χ_2^2. We find that $\chi_S = \chi_0 + \chi_2 + \chi_6$ and χ_9 is a new irreducible.

χ_9	55	−1	−5	1	1	3	−1	0	−1	1	1	−1	0	0	0

We will now check the inner product of every restricted character and χ_V with the known irreducible characters of M_{12}.

Induced Character	χ_0	χ_1	χ_2	χ_5	χ_6	χ_8	χ_9	χ_{12}
$\chi_{(9,3)}$	0	0	0	0	0	0	0	0
χ_{λ_A}	0	0	0	1	0	0	0	1
$\chi_{(8,4)}$	0	0	1	0	1	0	0	0
$\chi_{(7,5)}$	0	1	0	0	1	0	0	0
$\chi_{(3,2,1)}$	0	0	0	0	0	0	0	0
χ_{λ_B}	0	0	0	0	0	0	0	1
χ_V	0	0	0	1	0	0	0	1

Define the following characters:

$$\chi_C = \chi_{(9,3)}$$
$$\chi_D = \chi_{(7,5)} - \chi_1 - \chi_6$$
$$\chi_E = \chi_{(3,2,1)}$$
$$\chi_F = \chi_{\lambda_B} - \chi_{12}$$
$$\chi_W = \chi_V - \chi_5 - \chi_{12}$$

The values of these are as follows.

M_{12}	1^{12}	2^4	2^6	3^3	3^4	4^2	2^24^2	5^2	$2^13^16^1$	6^2	2^18^1	4^18^1	2^110^1	11^1_A	11^1_B
χ_C	154	10	−6	1	4	−2	−2	−1	1	0	0	0	−1	0	0
χ_D	231	7	−9	−1	0	−1	−1	1	1	0	−1	−1	1	0	0
χ_E	320	0	0	−4	−4	0	0	0	0	0	0	0	0	1	1
χ_F	210	2	10	3	−3	−2	−2	0	1	−1	0	0	0	1	1
χ_W	485	5	5	−1	−1	−3	−3	0	−1	−1	1	1	0	1	1

Taking the inner products of each of these new characters with each other gives the following:

	χ_C	χ_D	χ_E	χ_F	χ_W
χ_C	2	1	0	0	1
χ_D	1	2	1	1	1
χ_E	0	1	2	1	2
χ_F	0	1	1	2	2
χ_W	1	1	2	2	4

Observe that χ_V has one constituent character in common with both χ_C and χ_D. Moreover, χ_C and χ_D have one constituent in common with each other. There are two possibilities, either χ_V, χ_C and χ_D have exactly one constituent in common, or there are no constituent characters in common to all three of them. Also note that the constituents of χ_E and χ_F are constituents of χ_W.

Let $\chi_U = \chi_C + \chi_D + \chi_E + \chi_F$ and let $\chi_Z = \chi_U - \chi_W$. Now, if χ_V, χ_C and χ_D have no irreducible characters in common to all three of them, then $\frac{1}{2}\chi_Z$ will be irreducible.

We can deduce that they have no irreducible characters in common to all three of them by considering that χ_D and χ_E have one irreducible character in common, but χ_C and χ_E do not. Therefore, the constituent that χ_C and χ_W have in common cannot be the same as the constituent χ_D, χ_E and χ_W have in common.

χ_U	595	19	−5	1	1	−5	−5	0	1	1	−1	−1	0	1	1
χ_W	485	5	5	−1	−1	−3	−3	0	−1	−1	1	1	0	1	1
χ_Z	110	14	−10	2	2	−2	−2	0	2	2	−2	−2	0	0	0
$\frac{1}{2}\chi_Z$	55	7	−5	1	1	−1	−1	0	1	1	−1	−1	0	0	0

We verify that $\langle \frac{1}{2}\chi_Z, \frac{1}{2}\chi_Z \rangle = 1$. Let $\chi_6 = \frac{1}{2}\chi_Z$, $\chi_{11} = \chi_C - \chi_6$, $\chi_{14} = \chi_D - \chi_6$, $\chi_{13} = \chi_E - \chi_{14}$ and $\chi_{10} = \chi_F - \chi_{13}$. It can be verified using the inner product that all of these new characters are irreducible. Now, recall χ_B. We find that $\langle \chi_B, \chi_{14} \rangle = 1$, define $\chi_3 = \chi_B - \chi_{14}$. Let χ_4 be the complex conjugate of χ_3, then we see that χ_3 and χ_4 are irreducible. The full character table is given in Table 8.4.

M_{12}	1^{12}	2^4	2^6	3^3	3^4	4^2	$2^2 4^2$	5^2	$2^1 3^1 6^1$	6^2	$2^1 8^1$	$4^1 8^1$	$2^1 10^1$	11^1_A	11^1_B
χ_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_1	11	3	-1	2	-1	3	-1	1	0	-1	1	-1	-1	0	0
χ_2	11	3	-1	2	-1	-1	3	1	0	-1	-1	1	-1	0	0
χ_3	16	0	4	-2	1	0	0	1	0	1	0	0	-1	ω	$\overline{\omega}$
χ_4	16	0	4	-2	1	0	0	1	0	1	0	0	-1	$\overline{\omega}$	ω
χ_5	45	-3	5	0	3	1	1	0	0	-1	-1	-1	0	1	1
χ_6	54	6	6	0	0	2	2	-1	0	0	0	0	1	-1	-1
χ_7	55	7	-5	1	1	-1	-1	0	1	1	-1	-1	0	0	0
χ_8	55	-1	-5	1	1	-1	3	0	-1	1	-1	1	0	0	0
χ_9	55	-1	-5	1	1	3	-1	0	-1	1	1	-1	0	0	0
χ_{10}	66	2	6	3	0	-2	-2	1	-1	0	0	0	1	0	0
χ_{11}	99	3	-1	0	3	-1	-1	-1	0	-1	1	1	-1	0	0
χ_{12}	120	-8	0	3	0	0	0	0	1	0	0	0	0	-1	-1
χ_{13}	144	0	4	0	-3	0	0	-1	0	1	0	0	-1	1	1
χ_{14}	176	0	-4	-4	-1	0	0	1	0	-1	0	0	1	0	0

Table 8.4: The character table of M_{12}, where $\omega = \frac{1}{2}(-1 + \sqrt{-11})$.

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