THE WEAKLY SIGN SYMMETRIC $\mathbf{P}_{0,1}^{+}$-MATRIX COMPLETION PROBLEM

BY

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Declaration

Declaration by the candidate

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Dedication

I wound like to dedicate this research to my dear father Joshua Tomno Kipkochoi and mother Lina Tomno, who laid a good foundation for my initial education and supported me financially; and my beloved siblings for their love, continuous prayers and encouragement.

Acknowledgments

This thesis has been developed through a collaborative effort. I wish to express my sincere gratitude to all those who made valuable contributions in the development of this thesis.

Sincerely speaking, the set of people whom to acknowledge has roughly the cardinality of $\mathbb{N}$, making it impossible to list everyone. I will just list a subset of people to acknowledge, and those not mention are equally important and therefore nobody should feel offended if not listed in the subset.

{Dr.Ian Short, Dr.Fredrick Nyamwala, Dr.Kamaku Waweru, Dr.Nick Gill}

This subset is unordered, the names are listed alphabetically.

Dr.Ian Short is highly appreciated and commended for his constructive comments and suggestions that lead to the enhancement of this thesis. I am lucky to have his support in writing my thesis, despite not listing him as my supervisor, I can say confidently that I received focused supervision from him. Dr. Fredrick Nyamwala who I feel very fortunate and proud to have been his student in postgraduate classes and my supervisor in thesis writing is acknowledged for his support and guidance. Dr. Kamaku Waweru, who suggested an interesting and challenging problem is highly appreciated for his supervision and encouragement right from my undergraduate studies. I am also grateful to Dr.Nick Gill, who visited Kenya, June 2017 and July 2018. Dr. Gill's visits were very enjoyable and helpful to me. Though, I have not listed him as my supervisor, he contributed significantly to my thesis writing, both in research writing techniques and learning LaTex application for documentation of my thesis, I would like to express my gratitude to him.

To all those in the above set, I can only but say "thank you".

Abstract

A **weakly sign symmetric $\mathbf{P}^+_{0,1}$-matrix** is a weakly sign symmetric P^+_0-matrix with positive diagonal entries. A digraph D is said to have **weakly sign symmetric $P^+_{0,1}$-matrix completion** if every partial weakly sign symmetric $P^+_{0,1}$-matrix specifying D can be completed to a weakly sign symmetric $P^+_{0,1}$-matrix. Previous research has considered matrix completions for classes of matrices like P-matrices, P_0-matrices and $P_{0,1}$-matrices and their subclasses with little attention done on weakly sign symmetric $P^+_{0,1}$-matrices. In this research, a graph theoretic approach is used to achieve the necessary conditions for a digraph (or pattern) to have weakly sign symmetric $P^+_{0,1}$-matrix completion , classifying patterns associated with partial weakly sign symmetric P^+_0-matrices, partial weakly sign symmetric $P^+_{0,1}$-matrices and patterns having zero completion to a weakly sign symmetric $P^+_{0,1}$-matrices. The research will benefit computer engineers, statisticians and scientists in solving molecular conformation problems. It is shown that any asymmetric pattern associated with digraph of at most order 4 having weakly sign symmetric P-completion also have zero completion to a weakly sign symmetric $P^+_{0,1}$-matrix and patterns having weakly sign symmetric $P^+_{0,1}$-completion is completely classified for digraphs of at most order 3 and 192 out of 218 digraphs of order 4, the remaining patterns of order 4 are $q = 4, n = 1-2$; $q = 5, n = 1-5, 7, 17, 21$; $q = 6, n = 1, 3-8, 13, 15, 17, 19, 27, 38-39$ and $q = 7, n = 2, 9$. We, therefore, recommend for an investigation to the 26 unclassified patterns, that is, to know whether they are having weakly sign symmetric $P^+_{0,1}$-completion or not having weakly sign symmetric $P^+_{0,1}$-completion.

Notations and Terminologies

d_i:	Specified diagonal entry
a_{ij}:	Specified non-diagonal entry
x_{ij} :	Unspecified non-diagonal entry
c_{ij} :	Value assigned to unspecified non-diagonal entry x_{ij}
A:	$n \times n$ matrix
A_c:	Completed matrix
$\det A$:	Determinant of A
N:	Set of numbers $\{1, \ldots, n\}$
Π:	Class of matrices
$A(\alpha)$:	Principal sub-matrix where α is a subset of N
$\det A(\alpha)$:	Determinant of a principal sub-matrix $A(\alpha)$
G:	Graph
D:	Digraph
Q:	Pattern of a matrix
p:	Number of vertices of a digraph
q:	Number of arc(s) of a digraph
q:	Number of arc(s) of a digraph
wss :	Weakly sign symmetric
ss :	Sign symmetric

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Chapter 1

Introduction

This chapter presents a background of the study, definitions of terms used in this study, statement of the problem, justification, study objectives and finally, significance of the study.

1.1 Background of the Study

This section provides an overview concept of this study. Let us begin by briefly explaining our intentions with this study. Full definitions of all terms will be given in the next section.

A real $n \times n$ matrix A is a *P-matrix* (*P_0-matrix*) if, for each $k \in \{1, 2, \ldots, n\}$, every $k \times k$ principal minor of A is positive(nonnegative). A *weakly sign symmetric P-matrix* (*weakly sign symmetric P_0-matrix*) is a P-matrix (P_0-matrix) such that the product of a_{ij} and a_{ji} is nonnegative for all $i < j$.

A pattern Q has *P-completion (resp. P_0-completion)* if every partial P-matrix (resp. partial P_0-matrix) which specifies the pattern can be completed to a P-matrix (resp. P_0-matrix) as stated in (DeAlba & Hogben, 2000) and (Choi, DeAlba, Hogben, Maxwell, & Wangsness, 2002). The study we have just cited investigated completion for the P and P_0-classes of matrices.

The weakly sign symmetric property was later introduced in (DeAlba, Hardy, Hogben, & Wangsness, 2003) to classes of P-matrices and P_0-matrices giving rise to two subclasses of weakly sign symmetric P-matrices and weakly sign symmetric P_0-matrices.

In 2015, a new class of matrices: $P_{0,1}^{+}$-matrices was studied by (Sinha, 2017a). Progressive research has been done on matrix completions for the this class. In $P_{0,1}^{+}$-matrices, the weakly sign symmetric case have not been considered hence it is impressive to have a study that examines weakly sign symmetric $P_{0,1}^+$-completion.

This investigation can also be seen from another perspective: (DeAlba et al., 2003) studied completions of patterns of partial weakly sign symmetric P_0-matrices and partial weakly sign symmetric P-matrices which are matrices with properties that all principal minors are nonnegative and positive respectively. The case where all principal minors are nonnegative and at least one is positive in each order was not considered, these are the partial weakly sign symmetric P_0^+-matrices. In some sense, the class of weakly sign symmetric P_0^+-matrices lies half-way between the class of weakly sign symmetric P_0-matrices, and the class of weakly sign symmetric P_0-matrices. Our aim it to use the above study to derive a classification of patterns for weakly sign symmetric $P_{0,1}^+$-matrices which is a subclass of weakly sign symmetric P_0^+-matrix.

Our interest in this study is to understand when a partial weakly sign symmetric $P_{0,1}^+$-matrix can be completed. More precisely, we have investigated patterns Q having the property that all partial weakly sign symmetric $P_{0,1}^+$-matrices specified by pattern Q are completable. This problem have been addressed through application of matrix completion techniques.

1.2 Definition of Terms

In this section we define the basic concepts in linear algebra, group theory and graph theory that are commonly used and are fundamental in matrix completion problems.

The definition of terms related to classes of matrices are covered in §1.2.1; partial matrices are covered in §1.2.2; graphs, digraphs and patterns are covered in §1.2.3; and matrix completion is covered in §1.2.4.

1.2.1 Classes of matrices

In this subsection we define a large number of different classes of matrices. The formal definitions are given first, however the reader may gain better understanding by jumping forward to Figure 1.1 which connects all of the different definitions, and gives an easy way to understand these definitions using five basic properties.

Definition 1.2.1.

A *matrix* A is an ordered set of elements listed in a rectangular array, i.e., $m \times n$ where m is the number of rows and n is the number of columns. A *sub-matrix* of a matrix A is a smaller matrix obtained by deleting some row(s) and/or column(s) from matrix A. A *square matrix* A is a matrix that has n rows and n columns i.e., an $n \times n$ matrix. A *diagonal matrix* A is a matrix with all non-diagonal elements as zero i.e., if $i \neq j$ then $a_{ij} = 0$.

Definition 1.2.2.

For α a subset of $\{1, 2, ..., n\}$, the *principal sub-matrix* $A(\alpha)$ is obtained from $n \times n$ matrix A by deleting all rows and columns not indexed by an element of α. A *principal minor* of A is the determinant of a principal sub-matrix of A (Choi et al., 2002).

The next definition gives different types of matrices with respect to the values accepted by their principal minors, and this will give us the classes of matrices.

Definition 1.2.3.

A P-*matrix* (P_0-*matrix*) is a matrix in which every principal minor of the matrix is positive (nonnegative) (Choi et al., 2002). A $P_{0,1}$-*matrix* is a P_0-matrix for which all diagonal entries are positive (Choi et al., 2003). A real $n \times n$ matrix A is a P_0^+-*matrix* if for each $k \in$\{1,2,. . . ,n\}, every $k \times k$ principal minor of A is nonnegative and at least one $k \times k$ principal minor is positive. A P_0^+-matrix with positive diagonal entries is called a $P_{0,1}^+$-*matrix* as defined in (Sarma & Sinha, 2015a) and (Sinha, 2017a).

For more understating of the matrices defined above, we give Example 1.2.4 for elaboration.

Example 1.2.4.

Consider the following 2×2 matrices.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Matrices A, B, C and D are P_0-matrices since every principal minor for each matrix is nonnegative, and matrices B and D are $P_{0,1}$-matrix.

Matrices C and D are P_0^+-matrices. Finally, observe that matrix D is the only P-matrix and $P_{0,1}^+$-matrix.

Apart form looking at the principal minors, it is also very important to consider the types of entries which gives more restriction for each class of matrices, this will give us different subclasses. The four restrictions are given in the next definitions.

Definition 1.2.5.

A $n \times n$ matrix $A = [a_{ij}]$ is a

i *Weakly sign symmetric matrix* if $a_{ij}a_{ji} \geq 0$ for all i, j.

ii *Sign symmetric matrix* if $a_{ij}a_{ji} > 0$ or $a_{ij} = a_{ji} = 0$ for all i, j.

iii *Nonnegative matrix* if $a_{ij} \geq 0$ for all i, j.

iv *Positive matrix* if $a_{ij} > 0$ for all i, j.

Using Definition 1.2.5, we have four different subclasses for each class that we have defined in Definition 1.2.3. In this study, we have used the class of $P_{0,1}^+$-matrices to give the definitions for various subclasses after incorporating the four restrictions, other classes we will summarize using Figure 1.1.

To make our definitions more simpler here, we will relate with the classes of matrices that we have already defined instead of stating all other properties that have been inherited by the new subclass. For our case here, we have used $P_{0,1}^+$-matrix.

Definition 1.2.6.

A *weakly sign symmetric $P_{0,1}^{+}$-matrix* is a $P_{0,1}^{+}$-matrix whose product of twin entries is nonnegative, that is $a_{ij}a_{ji} \geq 0$ for all i and j. A *sign symmetric $P_{0,1}^{+}$-matrix* is a $P_{0,1}^{+}$-matrix whose product of twin entries is positive or both zeros, that is $a_{ij}a_{ji} > 0$ or $a_{ij} = a_{ji} = 0$ for all i and j. A *nonnegative $P_{0,1}^{+}$-matrix* is a $P_{0,1}^{+}$-matrix whose entries are nonnegative i.e., $a_{ij} \geq 0$ for all i and j. A *positive $P_{0,1}^{+}$-matrix* is a $P_{0,1}^{+}$-matrix whose entries are positive i.e., $a_{ij} > 0$ for all i and j.

Example 1.2.7.

The matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -3 & 4 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ is a weakly sign symmetric $P_{0,1}^{+}$-matrix but not a sign symmetric $P_{0,1}^{+}$-matrix.

First, all the diagonal entries are positive.

Second, we show that all principal minors are nonnegative and at least one in each order is positive.

$$\begin{aligned} \det A(1) &= 6; & \det A(1,2) &= 18; \\ \det A(2) &= 4; & \det A(1,3) &= 12; \\ \det A(2) &= 2; & \det A(2,3) &= 6; \\ \det A &= 12. \end{aligned}$$

The calculations shows that A is $P_{0,1}^{+}$-matrix, satisfying conditions of the principal minors.

Now since $a_{ij}a_{ji} \geq 0$ for all i and j then A is a weakly sign symmetric $P_{0,1}^{+}$-matrix and it is not a sign symmetric $P_{0,1}^{+}$-matrix because $a_{13} = 2 \neq a_{31} = 0$.

It is also clear that A is neither a nonnegative $P_{0,1}^{+}$-matrix nor a positive $P_{0,1}^{+}$-matrix due to existence of negative entries in the matrix.

The matrix definitions that we have encountered so far are summarized in Figure 1.1.

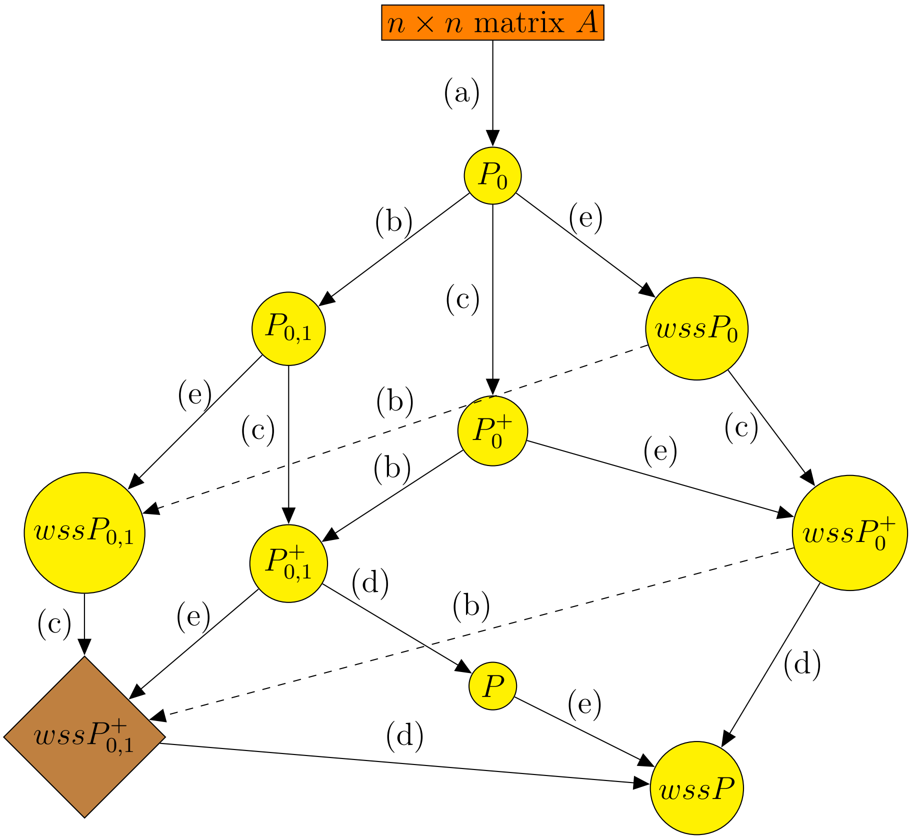

Figure 1.1: Different classes of matrices and their defining properties

The nodes of the figure correspond to different classes of matrices discussed above, while the edge labels correspond to the following properties for an n-by-n matrix A:

(a) all principal minors of A are non-negative;

(b) all diagonal entries of A are positive;

(c) for all $k = 1, \ldots, n$, at least one principal minor of A of order k is positive;

(d) all principal minors of A are positive;

(e) $a_{ij}a_{ji} \geq 0$ for all $i, j = 1, \ldots, n$.

To understand the diagram, an example will suffice. A matrix is *weakly sign symmetric (wss)* $P_{0,1}^+$ if it satisfies all of the conditions on a path from the top of the diagram to the node $wssP_{0,1}^+$. There are 6 paths to this node with labels:

1. (a), (b), (e), (c);
2. (a), (b), (c), (e);
3. (a), (c), (b), (e);
4. (a), (c), (e), (b);
5. (a), (e), (b), (c);
6. (a), (e), (c), (b).

The 6 sets of conditions are entirely equivalent and either suffices to define this class of matrices.

The diagram also makes it clear which classes of matrices contain which others: for instance, the class of P_0^+-matrices contain the class of P-matrices because the node P_0^+ occurs along a path to node P.

The diamond node correspond to the matrix–class that we are interested in this research.

Note that for the other three cases of entries affects only the last path (e) when we replace nodes with wss in Figure 1.1 as follows:

1. If sign symmetric(ss) then set (e) to $a_{ij}a_{ji} > 0$ or $a_{ij} = a_{ji} = 0$ for all $i, j = 1, \dots, n$.
2. If nonnegative then set (e) to $a_{ij} \geq 0$ for all $i, j = 1, \dots, n$.
3. If positive then set (e) to $a_{ij} > 0$ for all $i, j = 1, \dots, n$.

These replacement completes definitions of various classes of P-matrices.

All through we have been defining matrices with all entries specified, in the next subsection we will be defining matrices with some unspecified entries.

1.2.2 Partial matrices

In this subsection we define partial matrices for various different classes of matrix.

Definition 1.2.8.

A *partial matrix* is a matrix in which some entries are specified while others are free to be chosen from a certain set (DeAlba, Hogben, & Sarma, 2009).

We wish to extend this definition to deal with certain classes of matrices. Roughly speaking, given a class of matrices Π such as P_0-matrices, nonnegative

P_0-matrices, weakly sign symmetric P_0^+-matrices and so on, a *partial* Π-*matrix* is one whose specified entries satisfy the required properties of a Π-matrix.

Let us make this precise for three important classes of matrices:

Definition 1.2.9.

1. A *partial weakly sign symmetric* P_0-*matrix* is a partial matrix A such that the determinants of all fully specified principal sub-matrices are nonnegative and $a_{ij}a_{ji} \geq 0$ for all specified entries (DeAlba et al., 2003).

2. A *partial* P_0^+-*matrix* is a partial matrix A in which all fully specified principal minors are nonnegative and $S_k(A) > 0$ for every $k \in \{1, 2, ..., n\}$, whenever all $k \times k$ principal sub-matrices are fully specified (Sarma & Sinha, 2015a).

3. A *partial weakly sign symmetric* P_0^+-*matrix* is a partial P_0^+-matrix whose products of specified twin entries are nonnegative and a *partial weakly sign symmetric* $P_{0,1}^+$-*matrix* is a partial weakly sign symmetric P_0^+-matrix with positive diagonal entries.

The example below illustrates much about partial matrices.

Example 1.2.10.

The real matrix $A = \begin{bmatrix} 3 & 1 & -2 \\ 3 & 2 & x \\ -1 & y & 4 \end{bmatrix}$ is a partial weakly sign symmetric P_0^+-matrix since the determinants of all fully specified principal sub-matrices are nonnegative and $a_{ij}a_{ji} \geq 0$ for all specified entries. It is also a partial weakly sign symmetric $P_{0,1}^+$-matrix since all diagonal entries are positive.

By the fact that weakly sign symmetric P_0^+-matrix carries more restrictions than both weakly sign symmetric P_0-matrix and P_0^+-matrix then matrix A is weakly sign symmetric P_0-matrix and P_0^+-matrix.

It is clear from (3) of Definition 1.2.9 that the classes of partial weakly sign symmetric $P_{0,1}^+$-matrices splits into three cases:

1. At least one diagonal entries is unspecified;

2. All diagonal entries are specified, and at least one non-diagonal entry is unspecified;

3. All entries of A are specified.

Example 1.2.10 gives the second case of the partial weakly sign symmetric $P_{0,1}^{+}$-matrices.

Our study focused on the second and third case, where all the diagonal entries are specified and this is because we have used patterns associated with digraphs, more information has been given in the next subsection.

Later, we represented entry positions of matrices using digraphs (or patterns) and hence the next subsection on digraphs and patterns.

1.2.3 Graphs, digraphs and patterns

In this subsection we define graph and pattern terminologies which are important in matrix completion.

Definition 1.2.11.

A *graph* $G = (V_G, E_G)$ is a finite non-empty set of positive integers V_G, whose members are called *vertices* and a set, E_G, of (unordered) pairs $\{u,v\}$ of vertices called the *edges* of G. Given a graph $G = (V_G, E_G)$ then a graph $H = (V_H, E_H)$ is a *subgraph* of graph G if V_H is a subset of V_G and E_H is a subset of E_G. A graph whose edge-set is empty is a *null graph.*

The graph definition here differs from standard use in that we require vertices to be positive integers (since we will be using them to represent matrices) as defined in (Hogben, 2001). An example of a graph and a subgraph is given in Figure 1.2.

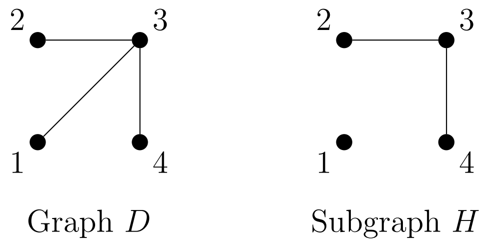

Figure 1.2: Graph D and a subgraph H

Definition 1.2.12.

A *digraph* $D = (V_D, E_D)$ is a finite non-empty set of positive integers V_D, whose members are called *vertices* and a set, E_D, of (ordered) pairs (u,v) of vertices called the *arc* of D. Given an arc $x = (u, v)$, the vertex u is called the *initial* vertex (tail); v is the *terminal* vertex (head); and we say that x is *adjacent* to u and v, or equivalently, we say that u is *adjacent to* v. An arc joining a vertex to itself is called a *loop* i.e. $x = (v, u)$ and $v = u$. A digraph $H = (V_H, E_H)$ is a *sub-digraph* of digraph D if $V_H \subseteq V_D$ and $E_H \subseteq E_D$ (Harary, 1969). An example of a digraph is given in Figure 1.3; it has an arc (1,2) with initial vertex 1 and terminal vertex 2.

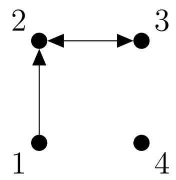

Figure 1.3: Digraph of order 4 and 3 arcs

Note that the *underlying* graph G of a digraph D is the graph obtained by replacing each arc (i, j) or pair of arcs (i, j) and (j, i) if both are present by the one edge {i, j}. Arc (i, j) (or arcs (i, j) and (j, i) if both are present) of D and edge {i, j} of G are said to *correspond*.

There are two important types of sub-digraphs and are given below.

Definition 1.2.13.

Let D be a digraph, then

1. The sub-digraph H is an *induced sub-digraph (induced by V_H)* if, for every $u, v \in V_H$ such that the arc $(u, v) \in E_D$, we have $(u, v) \in E_H$. If v is a vertex of D, we write $D - v$ for the subgraph induced by $D_D \setminus \{v\}$.

2. The sub-digraph H is a *spanning sub-digraph* if V_H=V_D, i.e. only arcs are deleted.

Definition 1.2.14.

The *order* of a digraph D, denoted by $|D|$, is the number of vertices of D. A digraph is *complete* if it includes all possible arcs between its vertices, and is denoted by K_n, where n is the number of vertices. A complete (di)graph is called a *clique*.

The order of the digraph in Figure 1.3 is 4.

Definition 1.2.15.

A *path* P in a digraph $D = (V_D, E_D)$ is a sub-digraph of (V_P, E_P) where $V_P = \{v_1, \ldots, v_k\}$ and $E_P = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\}$.

In this case the *length* of P is $k - 1$, and we represent P as follows: $v_1 \to v_2 \to ... \to v_k$. A *cycle* C in a digraph $D = (V_D, E_D)$ is a sub-digraph (V_C, E_C) where $V_C = \{v_1, \ldots, v_k\}$ and $E_C = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1)\}$. in this case the *length* of C is k and we call C a k-cycle.

The digraph in Figure 1.3 contains one path of length 2: $1 \to 2 \to 3$ and contains no cycles.

Definition 1.2.16.

A *homomorphism* between two digraphs maps vertices to vertices, arcs to arcs while preserving the incidence relation. More precisely, a homomorphism ϕ between the digraph D and the digraph $D^{'}$ is a mapping $\phi : V_D \cup E_D \to V_{D'} \cup E_{D'}$ such that for each arc $a \in E_D$, $\phi(s(a)) = i(\phi(a))$ and $\phi(t(a)) = t(\phi(a))$ where i

and t are maps that assigns to each arc two elements of D: the initial vertex and the terminal vertex. A homomorphism $\phi : D \longrightarrow D'$ is an *isomorphism* if the homomorphism is bijective (Keidar, 2009).

Note that a digraph $D = (V_D, E_D)$ is *isomorphic* to the digraph $D' = (V_{D'}, E_{D'})$ if there is a bijection map $\phi : V_D \rightarrow V_{D'}$ such that $(v, w) \in E_D$ if and only if $(\phi(v), \phi(w)) \in E_{D'}$.

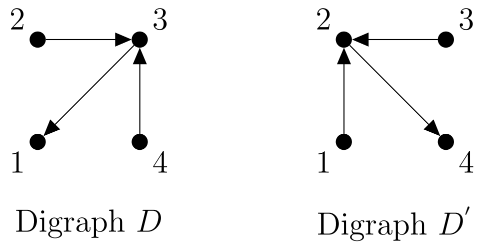

Figure 1.4: Digraph D and an isomorphic digraph D'

Consider digraph D and digraph D' in Figure 1.4 and a bijection map $\phi : V_D \rightarrow V_{D'}$ given by $v_1 \rightarrow v_4$, $v_2 \rightarrow v_1$, $v_3 \rightarrow v_2$ and $v_4 \rightarrow v_3$ then observe that $(v_i, v_j) \in E_D$ if and only if $(\phi(v_i), \phi(v_j)) \in E_D'$.

Definition 1.2.17.

A *pattern* Q for $n \times n$ partial matrices is a list of positions of the $n \times n$ matrix, that is subset of {1,...,n} $\times$ {1,...,n} that includes all diagonal positions. A *symmetric pattern* is a pattern with the property that (i, j) is in the pattern if and only if (j, i) is in the pattern. An *asymmetric pattern* is a pattern with the property that (i, j) is in the pattern, then (j, i) is not in the pattern. A partial matrix *specifies a pattern* if its specified entries lie exactly in those positions listed in the pattern (Choi et al., 2003).

We emphasize that in this research, all patterns contain diagonal entries since we will be using digraphs with specified vertices; and for that reason we have considered partial matrices with specified diagonal entries.

It is important to note that patterns and digraphs are interchangeable as illustrated in Example 1.2.18.

Example 1.2.18.

The 3×3 pattern $Q = \{(1,1),(1,2),(1,3),(2,2),(3,1),(3,3)\}$ is specified by the partial matrix $A = \begin{bmatrix} d_1 & a_{12} & a_{13} \\ x_{21} & d_2 & x_{23} \\ a_{31} & x_{32} & d_3 \end{bmatrix}$.

Note that the pattern Q is neither a symmetric pattern nor an asymmetric pattern. Pattern Q is represented as a digraph of 3 vertices and 3 arcs given by Figure 1.5.

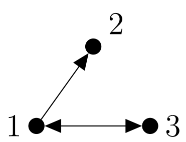

Figure 1.5: Digraph of order 3 and 3 arcs

Definition 1.2.19.

A pattern Q is *permutation similar* to a pattern R if there is a permutation ϕ of {1,...,n} such that $R = \{((\phi(i), \phi(j)) : (i,j) \in Q\}$.

Relabeling the vertices of a digraph diagram, which performs a digraph isomorphism, corresponds to performing a permutation similarity on the pattern (Hogben, 2003a), and since subclasses of $P_{0,1}^{+}$-matrices are closed under permutation similarity, then we require only to determine completions of non-isomorphic digraphs.

We have already given definitions of partial matrices and also digraphs, now in the next subsection we will define how to complete those partial matrices.

1.2.4 Matrix completion

In this subsection we give some definitions on matrix completion and later give the between completion of a partial matrix and a pattern (or a digraph).

Definition 1.2.20.

A *completion* of a partial matrix is a specific choice of values for the unspecified entries (Choi et al., 2003). Completion of a partial matrix is called *zero completion* if all the unspecified entries in the partial matrix are equated to zeros.

Definition 1.2.21.

A pattern has *weakly sign symmetric $P^+_{0,1}$-completion* if every partial weakly sign symmetric $P^+_{0,1}$-matrix that specifies the pattern can be completed to a weakly sign symmetric $P^+_{0,1}$-matrix.

In general, a pattern (or its (di)graph) has Π-completion if every partial Π-matrix which specifies the pattern can be completed to a Π-matrix.

Definition 1.2.22.

A pattern has *zero completion to a weakly sign symmetric $P^+_{0,1}$-matrix* if every partial weakly sign symmetric $P^+_{0,1}$-matrix that specifies the pattern can be completed to a weakly sign symmetric P^+_0-matrix by assigning all unspecified entries to zeros.

We wish to differentiate completion of a partial matrix and a pattern using Example 1.2.23 and Example 1.2.24.

Example 1.2.23.

The partial weakly sign symmetric $P^+_{0,1}$-matrix $A = \begin{bmatrix} 3 & 2 & -3 \\ 1 & 4 & x_{23} \\ -2 & -1 & 2 \end{bmatrix}$ specifies pattern $Q = \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1),(3,2),(3,3)\}$.

The determinants of all fully specified principal sub-matrix are nonnegative and $a_{ij}a_{ji} \geq 0$ for all specified entries.

Note that it is not a require for a partial matrix with at least one unspecified entry to have a positive principal minor.

If we analyze the principal sub-matrices of $A(2,3)$, the entry x_{23} must be negative so that the condition $a_{ij}a_{ji} \geq 0$ is satisfied.

The determinant of A i.e., $\det A = 3(8 + x_{23}) - 2(2 + 2x_{23}) - 3(-1 + 8) = 24 + 3x_{23} - 4 - 4x_{23} - 21 = -1 - x_{23} > 0$; this implies that $x_{23} < -1$ because the determinant needs to be positive.

The determinant of sub-matrix $A(2,3)$ is given by $\det A(2,3) = 8 + x_{23} \geq 0$ implying that $x_{23} \geq -8$. (At least $A(1,2)$ is positive.

The intersection of the two sets is $-8 \leq x_{23} < -1$ which satisfies the conditions of a weakly sign symmetric $P_{0,1}^{+}$-matrix. Hence the above partial weakly sign symmetric $P_{0,1}^{+}$-matrix can be completed to has weakly sign symmetric $P_{0,1}^{+}$-matrix by set $-8 \leq x_{23} < -1$.

Now the fact that, the partial matrix has been completed to weakly sign symmetric $P_{0,1}^{+}$-matrix does not mean that the pattern Q has weakly sign symmetric $P_{0,1}^{+}$-completion.

In that case we give an example to show that the pattern Q does not have weakly sign symmetric P_{0}^{+}-completion.

Example 1.2.24.

The partial weakly sign symmetric P_{0}^{+}-matrix $A = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 1 & x_{23} \\ 0 & 0 & 2 \end{bmatrix}$ specifies pattern

$$Q = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1), (3,2), (3,3)\}.$$

The determinant of matrix A is zero, i.e., $\det A = 0$ for any choice of x_{23}; therefore it can not be completed to a weakly sign symmetric P_{0}^{+}-matrix. Hence pattern Q does not have weakly sign symmetric P_{0}^{+}-completion.

Note that, completion of digraphs (or pattern) is more powerful since it considers the completion of every partial matrix specifying the pattern. The existence of a partial matrix that can not be completed causes lack of completion of a pattern.

1.3 Statement of the Problem

There has been a lot of research on the matrix completion problems for patterns with respect to several classes of matrices; we mention in particular, P-matrices (Bowers et al., 2006), P_0-matrices (Choi et al., 2002), $P_{0,1}$-matrices (Wangsness, 2005), P_0^+-matrices (Sarma & Sinha, 2015a) and $P_{0,1}^+$-matrices (Sarma & Sinha, 2015a).

The subclasses of P_0-matrices and P-matrices were later researched, these are nonnegative P_0-matrices in (Choi et al., 2003) and (weakly) sign symmetric P-matrices in (DeAlba et al., 2003). At the least, in all these research works, digraphs of up to 4 vertices were considered.

In this study we investigated matrix completion of weakly sign symmetric $P_{0,1}^+$-matrices. A major difficulty with these classes of matrix is that they lack the so-called "hereditary property", i.e., a matrix A can belong to one of these classes, while one of its principal sub-matrices does not. This makes the analysis much harder. Many of the classes listed above – for example the P-matrices, P_0-matrices and $P_{0,1}$-matrices and their subclasses considering the entries **do** have this property and so the analysis of matrix completion here is more straightforward.

By the way, referring to Figure 1.1, we see that the node $wssP_{0,1}^+$ is sandwiched between nodes $wssP_{0,1}$ and $wssP$ – these latter nodes correspond to classes that have the hereditary property, while the former does not. It is not always immediately clear which classes possess the hereditary property.

Nonetheless, some of the classes listed above also lack the hereditary property. In particular, we mention the class of P_0^+-matrices and the class of $P_{0,1}^+$-matrices; these have been studied by (Sarma & Sinha, 2015a) and (Sinha, 2017a), respectively. We make use of the analysis in these references to inform our own analysis of the two new classes.

Indeed, we go further: in (Wangsness, 2005), Wangsness investigated the completion problem for weakly sign symmetric $P_{0,1}$-matrices using the investigation for weakly sign symmetric P-matrices and weakly sign symmetric P_0-matrices found in (DeAlba et al., 2003). Similarly, this study on completion problem for weakly sign symmetric $P_{0,1}^{+}$-matrices makes use of the investigations on weakly sign symmetric P-matrices found in (DeAlba et al., 2003).

1.4 Justification of the Study

Situations arise when a full set of data is not available or is not economical to collect. With the knowledge that the complete data set must have certain properties when arranged in a matrix, the values of the unavailable data can be suggested.

According to Hogben (Hogben, 2001), extensive research has been done on matrix completion for classes including P-matrices, P_0-matrices, nonnegative P-matrices, (weakly) sign symmetric P-matrices, (weakly) sign symmetric P_0-matrices and most recently, Sarma researched on P_0^{+}-matrices in (Sarma & Sinha, 2015a) and and Sinha researched on $P_{0,1}^{+}$-matrices in (Sinha, 2017a), matrix completion research regarding various classes of $P_{0,1}^{+}$-matrices has not been done, hence the research on weakly sign symmetric $P_{0,1}^{+}$-matrices, was necessary.

1.5 Objective of the Study

In this section we have given both general and specific objectives of this research.

1.5.1 General Objective

Our objectives concern the matrix completion problem for the class of weakly sign symmetric $P_{0,1}^{+}$-matrices.

1.5.2 Specific Objectives

The specific objectives of this study were to:-

(i) Determine the necessary conditions for a digraph (or pattern) to have weakly sign symmetric $P_{0,1}^{+}$-matrix completion.

The results of this objective is presented in Section §4.1.

(ii) Determine the relationship between weakly sign symmetric $P_{0,1}^{+}$-completion and other classes of weakly sign symmetric P-matrix.

The results of this objective is presented in Section §4.2.

(iii) Characterize patterns associated with digraphs of order at most 4 having weakly sign symmetric $P_{0,1}^{+}$-completion.

The results of this objective is presented in Section §4.3.

(iv) Single out patterns associated with digraphs of order at most 4 having zero completion to a weakly sign symmetric $P_{0,1}^{+}$-matrices.

The results of this objective is presented in Section §4.4.

1.6 Significance of the Study

Work on matrix completion can be applied in many areas where some information is known but other information is not available and it is known that the full data matrix must have certain properties.

Some of the areas where matrix completion is useful include computer engineering problems such as data transmission, coding, decompression and image enhancement, system theory, discrete optimization (relaxation method), statistical problems like the entropy method for missing data, chemistry problems like molecular conformation problems, operator theory, and also in geophysical problems like in seismic reconstruction problems as discussed in (Lee & Seol, 2001) and (Choi et al., 2003).

For instance, when an image file is transmitted across the internet, it is described as a matrix of hexadecimal entries each encoding the color to a particular pixel in the image. If this file is corrupted during transmission, then one may be left with a partial matrix which must be completed if the image is to be recovered. By placing restrictions on the form of the image matrices, using principle of coding theory, it is possible to perform this matrix completion efficiency and effectively.

Our work fits into the theoretical literature for this work of practical problem.

Chapter 2

Literature review

This chapter examines and acknowledges the contributions of other researchers and scholars on matrix completion. This is done through the review of books, journals, research work and electronic sources. Matrix completion form a major area of interest for mathematicians in abstract algebra.

Throughout this chapter and the next chapters, digraphs will be denoted as $D_p(q,n)$ where p denotes the number of vertices, q denotes the number of arcs and n denotes the diagram number given in (Harary, 1969). The serial number n is important in distinguishing non-isomorphic digraphs having the same number of vertices and arcs. Recall that, we write weakly sign symmetric and sign symmetric in short form as wss and ss respectively, e.g. $wssP_0^+$-matrix instead of weakly sign symmetric P_0^+-matrix.

We will give available literature of related works for the previous studies on completions for various classes of P-matrices, their relationships and finally close the chapter by identifying the gaps in the literature which will be addressed in this research, and are organized as follows:

Section §2.1: Completions for various classes of P-matrices.

Section §2.2: Relationship between various matrix completion.

Section §2.3: Gaps in the literature.

2.1 Completions for various classes of P-matrices

In this section we have given some results on completions for various classes of P-matrices which are organized as shown in Figure 2.1.

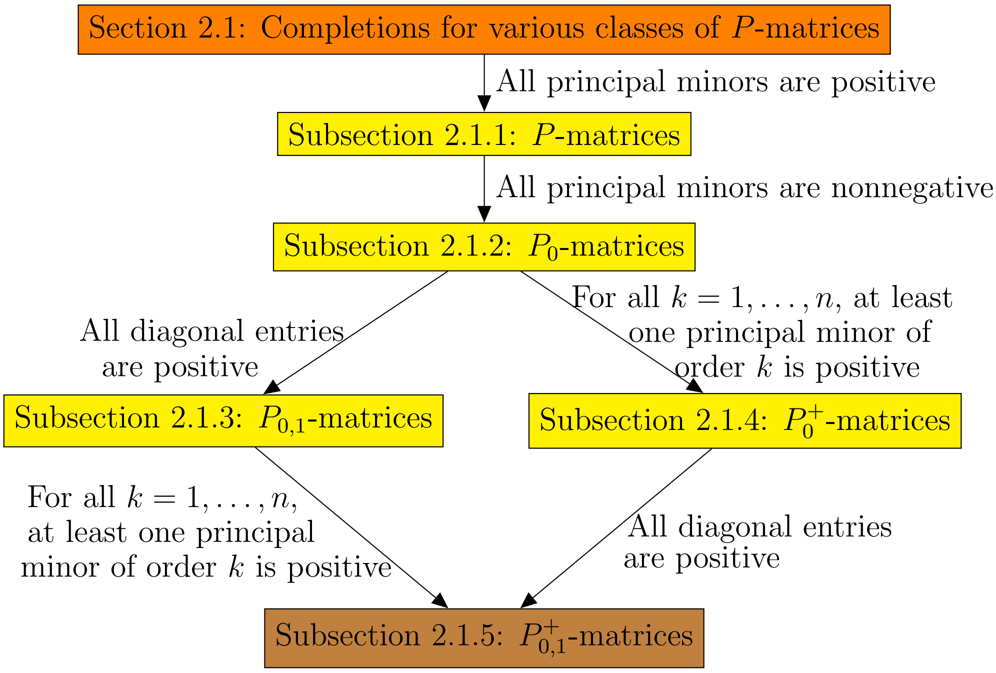

Figure 2.1: Completions for various classes of P-matrices

The path linking various classes is accompanied by a short description of the additional property leading to the formation of new class. It also shows the flow of informations in the literature.

In each of the subsections, we will consider 5 subclasses of the respective class, that is, π-matrices, wss π-matrices, ss π-matrices, nonnegative π-matrices and positive π-matrices where π is a class of P, P_0, $P_{0,1}$, P_0^+ and $P_{0,1}^+$.

The first class to be reviewed in the subsection below is the class of P-matrices, recall that P-matrix is a matrix in which every principal minor is positive.

2.1.1 P-matrix completion

The study of P-matrix completion was first introduced by Johnson and Kroschel, they studied combinatorially symmetric P-matrix completion (Johnson & Kroschel, 1996). It was proved first, that any symmetric pattern that contains the diagonal entries has P-completion although this does not hold for P_0-completion; second, all 3×3 patterns have P-completion; third, showed that for every $n \times n$ partial P-matrix with exactly one unspecified entry and $n \geq 4$, there is a partial P-matrix that lacks P-matrix completion.

DeAlba and Hogben researched on "completion of P-matrix patterns" in (DeAlba & Hogben, 2000); this is an extension of the work of Johnson and Kroschel. They classified digraphs of order 4 and it was found, $q = 1-8, 12$, $q = 9, n = 1, 2, 8, 11$; $q = 10, n = 1$ have P-completion and $q = 9, n = 3$; $q = 10, n = 5$ and $q = 11$ do not have P-completion. The classification was not complete since there were 11 unclassified patterns which include $q = 9, n = 4-7, 9, 10, 12, 13$.

The classification of digraphs of order less than or equal to 4 regarding (weakly) sign symmetric P-completion was done in (DeAlba et al., 2003). They found that, first, all digraphs of order 1 and 2 have (weakly) sign symmetric P-completion; second, digraph of order 3 has (weakly) sign symmetric P_0-completion if and only if its digraph does not contain a 3-cycle or is complete; and third, digraph of order 4 has (weakly) sign symmetric P-completion if and only if its digraph is one of the following.

$q = 0, \quad n = 1;$

$q = 1, \quad n = 1;$

$q = 2, \quad n = 1-5;$

$q = 3, \quad n = 1-11, 13;$

$q = 4, \quad n = 1-12, 14-19, 21-23, 25-27;$

$q = 5, \quad n = 1-5, 7-10, 14-17, 21-24, 26-29, 31, 33-34, 36-37$

$q = 6, \quad n = 1-8, 13, 15, 17, 19, 23, 26-27, 32, 35, 38-40, 43, 46;$

$q = 7, \quad n = 2, 4-5, 9, 14, 24, 29, 34, 36;$

$q = 8, \quad n = 1, 10, 12, 18;$

$q = 9, \quad n = 8, 11;$

$q = 12, \quad n = 1.$

Nonnegative P-completion and positive P-completion was first considered in

(Fallat, Johnson, Torregrosa, & Urbano, 2000), it was shown that the all digraphs of order 1, 2 and 3 have nonnegative P-completion and positive P-completion, and later in (Bowers et al., 2006), similar research was done for digraphs of order 4 and it was shown that the only digraphs of order 4 that do not have both nonnegative P-completion and positive P-completion are:

$$
\begin{aligned}
&q = 6, && n = 45; \\
&q = 7, && n = 30, 32, 33, 35, 38; \\
&q = 8, && n = 16, 17, 19, 20, 22 - 26; \\
&q = 9, && n = 3 - 7, 9, 10, 12, 13; \\
&q = 10, && n = 1 - 5; \\
&q = 11, && n = 1.
\end{aligned}
$$

All the 5 cases have been studied under this class P-matrices.

The second class to be reviewed in the subsection below is the class of P-matrices, recall that P_0-matrix is a matrix in which every principal minor is nonnegative, the difference from the class we have reviewed above is that it accepts zero as the principal minors (new property from the previous class of P-matrices).

2.1.2 P_0-matrix completion

The P_0-matrix completion problem was investigated in (Choi et al., 2002) and it was established that every asymmetric pattern has P_0-completion, all digraphs of order 1, 2 and 3 except $D_3(4, 2)$ and $D_3(5, 1)$ have P_0-completion. Furthermore, they characterized digraphs of order 4. It was shown that if a digraph of order 4 is listed below, then it has P_0-completion.

$q = 0, \qquad n = 1;$

$q = 1, \qquad n = 1;$

$q = 2, \qquad n = 1 - 5;$

$q = 3, \qquad n = 1 - 13;$

$q = 4, \qquad n = 1 - 12, 14 - 27;$

$q = 5, \qquad n = 1 - 5, 7 - 10, 14 - 17, 21 - 38$

$q = 6, \qquad n = 1 - 8, 13, 15, 17, 19, 23, 26 - 27, 32, 35, 38 - 40, 43, 45 - 48;$

$q = 7, \qquad n = 2, 4 - 5, 9, 14, 24, 29, 34, 36;$

$q = 8, \qquad n = 1, 10, 12, 18;$

$q = 9, \qquad n = 8, 11;$

$q = 12, \qquad n = 1.$

On the weakly sign symmetric P_0-completion, it was shown in (DeAlba et al., 2003), that all patterns weakly sign symmetric P-completion except $q = 4, n = 16$; $q = 5, n = 7$; $q = 6, n = 4, 7$; $q = 7, n = 2$ also have weakly sign symmetric P_0-completion for digraphs of at most order 4.

The nonnegative P_0-matrix completion problem was studied in (Choi et al., 2003). The study showed that all digraphs of order 1, 2 and 3 have nonnegative P_0-completion. They also examined digraphs of order 4 and found that those which do not have nonnegative P_0-completion are those digraphs that do not have nonnegative P-completion with additional digraphs $q = 4, n = 16$; $q = 5, n = 7, 32$; $q = 6, n = 4, 7, 22, 33, 34, 37, 42$; $q = 7, n = 2, 8, 10, 12, 13, 18, 20, 20, 25, 27$; $q = 8, n = 2 - 5, 7 - 9, 11, 13$; $q = 9, n = 1 - 2$.

Under this subclass of P_0-matrices, only 3 cases out of 5 have been studied. The cases that were not studied are for sign symmetric P_0-matrices and positive P_0-matrices.

The third class to be reviewed in the subsection below is the class of $P_{0,1}$-matrices. Recall that $P_{0,1}$-matrix is a matrix in which every principal minor is nonnegative and having positive diagonal entries (new property from the previous class of P_0-matrices).

2.1.3 $P_{0,1}$-matrix completion

Matrix completion problems regarding various classes of $P_{0,1}$-matrices was considered in the PhD thesis of Amy Lee Wangsness (Wangsness, 2005); the three subclasses that were studied by Wangsness are $P_{0,1}$-matrices, $wssP_{0,1}$-matrices and $ssP_{0,1}$-matrices. Her objectives were to classify digraphs of at most order 4. The relationships with other classes played a big role in her classifications. The results where as follows: First, a pattern has $P_{0,1}$-completion if and only if it is given in the list below:

$p = 1 - 3;$		
$p = 4;$	$q = 0 - 6, 12;$	
	$q = 7,$	$n = 1 - 29, 34, 36 - 37;$
	$q = 8,$	$n = 1 - 15, 18, 21, 27;$
	$q = 9,$	$n = 1 - 2, 11;$
	$q = 10,$	$n = 1.$

Secondly, she found that all patterns that have weakly sign symmetric $P_{0,1}$-completion are exactly those patterns that have weakly sign symmetric P-completion which have already been shown in Subsection 2.1.1.

Finally, in her other objective, she found that the only patterns with $ssP_{0,1}$-

completion are the ones listed below:

$p = 1 - 2;$		
$p = 3;$	$q = 0 - 2;$	
	$q = 3,$	$n = 1, 3 - 4;$
	$q = 4,$	$n = 1;$
$p = 4;$	$q = 0 - 2, 12;$	
	$q = 3,$	$n = 1 - 11, 13;$
	$q = 4,$	$n = 1 - 12, 16 - 19, 21 - 23, 25 - 27;$
	$q = 5,$	$n = 1 - 5, 7 - 10, 26 - 29, 31, 33 - 34, 36 - 37;$
	$q = 6,$	$n = 1 - 3, 46;$
	$q = 7,$	$n = 4 - 5;$
	$q = 8,$	$n = 1.$

From the available literature, it is noticed in this subsection that we lack literature for the case of nonnegative $P_{0,1}$-completion and positive $P_{0,1}$-completion.

The next two classes under subsection 2.1.4 and 2.1.5 are closely related to this study. They are different from the previous classes since hereditary property is not obeyed here, that is, if a digraph has P_0^+-matrix completion, it is not guarantee that its all sub-digraphs have P_0^+-matrix completion.

The fourth class to be reviewed in the subsection below is the class of P_0^+-matrices. It can not link directly with the immediate previous class $P_{0,1}$-matrices. We subtract the condition that all diagonal entries and add the new property that for all $k = 1, \ldots, n$, at least one principal minor of order k is positive or we rather link directly to class of P_0-matrices (see, Figure 2.1).

2.1.4 P_0^+-matrix completion

The study on P_0^+-matrix completion problem was done in (Sarma & Sinha, 2015a). They first presented two necessary conditions for P_0^+-matrix completion.

Secondly, they used the necessary conditions to give classifications of digraphs of order at most 4. The patterns that have P_0^+-completion are listed below.

$p = 1$		
$p = 2;$	$q = 0, 2;$	
$p = 3;$	$q = 0, 1, 6;$	
	$q = 2,$	$n = 2;$
$p = 4;$	$q = 0, 1, 12;$	
	$q = 2,$	$n = 1 - 5;$
	$q = 3,$	$n = 1 - 7, 9, 10, 12, 13;$
	$q = 4,$	$n = 1 - 9, 11, 16 - 20, 22 - 26;$
	$q = 5,$	$n = 1 - 3, 7 - 10, 25, 27, 30, 32, 33, 35, 38;$
	$q = 6,$	$n = 3 - 5, 7;$
$p = 12;$	$q = 1,$	$n = 1.$

Finally, they gave comparisons between P_0^+-completion and other classes.

The last class to be reviewed is the class of $P_{0,1}^+$-matrices, it is related to the class we have just reviewed above P_0^+-matrices. The additional property is that all the diagonal entries need to be positive. One can also see the connections with $P_{0,1}$-matrices by considering extra condition that for all $k = 1, \ldots, n$, at least one principal minor of order k is positive. The class of $P_{0,1}^+$-matrices can be approached in two ways via $P_{0,1}$-matrices or P_0^+-matrices as shown in Figure 2.1. This research is in this class and is the class with highest number of conditions compared to the set of classes that we have handled so far.

2.1.5 $P_{0,1}^+$-matrix completion

The $P_{0,1}^+$-matrix completion is the latest research to be studied out of all classes reviewed in the previous subsections. Sinha considered $P_{0,1}^+$-matrix completion

(Sinha, 2017a). The study was targeting classification of digraphs having order at most 4. He started his study by establishing necessary and some sufficient conditions. The results by Sihna showed that digraphs listed below have $P^{+}_{0,1}$-matrix completion:

$p = 1$	$q = 0,$	$n = 1;$
$p = 2;$	$q = 0, 2,$	$n = 1;$
$p = 3;$	$q = 0, 2, 3,$	$n = 1;$
	$q = 2, 3,$	$n = 1 - 4;$
	$q = 4,$	$n = 1, 2;$
	$q = 6,$	$n = 1;$
$p = 4;$	$q = 0, 2,$	$n = 1;$
	$q = 2,$	$n = 1 - 5;$
	$q = 4,$	$n = 10, 12 - 15;$
	$q = 5,$	$n = 4 - 6, 11 - 28;$
	$q = 6,$	$n = 1, 2, 6, 8, 8 - 23, 26, 28 - 30, 32 - 35, 37, 41, 42;$
	$q = 7,$	$n = 1 - 6, 8, 10, 14, 18, 24, 25, 27;$
	$q = 8,$	$n = 1, 2, 8, 11;$
	$q = 12,$	$n = 1.$

The given conditions were not strong enough to classify digraphs completely. Although most of the digraphs were classified, some remained unclassified. The list of 45 unclassified digraphs is given below:

$p = 4;$	$q = 6,$	$n = 24, 25, 27, 31, 36, 38 - 40, 44;$
	$q = 7,$	$n = 7, 9, 11 - 13, 15, 17, 19 - 21, 23, 26, 28, 30 - 33, 35, 37, 38;$
	$q = 8,$	$n = 3, 4, 6, 8, 9, 13, 16, 17, 19, 20, 23, 24, 25, 27;$
	$q = 9,$	$n = 4, 12.$

Any digraph not listed in any of the two lists above do not have $P^+_{0,1}$- completion.

He concluded his study by giving some relationships between $P^+_{0,1}$- completion and other classes.

In the next section, we give relationships between various matrix completions that have been studied in the classes we have reviewed. Most of the relationships connects classes of matrices with similar entries. The relationship among classes helps in knowing which set of digraphs to consider when doing classifications, hence at some point it acts as a necessary condition.

2.2 Relationship between various matrix completion

In 2003, Hogben analyzed related classes and their subclasses in (Hogben, 2003a). Let Π be P, $wssP$, ssP and nonnegative P, then the four very important relationships from her results are listed below:

1. Any digraph that has Π_0-completion also has Π-completion.

2. Any digraph that has Π_0-completion also has $\Pi_{0,1}$-completion.

3. Any digraph that has $\Pi_{0,1}$-completion also has Π-completion.

4. Asymmetric digraph that has Π-completion also has $\Pi_{0,1}$-completion.

In a separate paper, Hogben extended the study of relationships between pairs of classes in (Hogben, 2003b). She gave new result relating positive P-completion and negative P-completion. It was shown that, any pattern that has negative P-completion also has positive P-completion.

In the next section, we will be identifying gaps in literature as per the previous researches on classes of P-matrices.

2.3 Gaps in the literature

From the studies reviewed in Section §2.1 on completions for various classes of P-matrices, it is very evident that there are some subclasses that have not been studied. To make it clear, summary of the current state of knowledge for various classes of P-matrices is given in Table 2.1.

Table 2.1: Research works on completions for various classes of P-matrices

Entries / Class	**Undefined**	**Wss**	**Ss**	**Nonnegative**	**Positive**
P	Yes*	Yes	Yes	Yes	Yes
P_0	Yes	Yes	No	Yes	No
$P_{0,1}$	Yes	Yes	Yes	No	No
P_0^+	Yes	No	No	No	No
$P_{0,1}^+$	Yes*	No?	No	No	No

In the table above, "Yes" means it has been studied and complete classifications of digraphs of up to order 4 was given; "No" means there has been little attention on the class; an asterisk "*" by "Yes" means it has been studied with incomplete classification of digraphs of up to order 4; an asterisk "?" by "No" means it has been studied and it is where this research is based on.

By examining Table 2.1, three subclasses have been considered for weakly sign symmetric entries, that is, $wssP$-matrices, $wssP_0$-matrices and $wssP_{0,1}$-matrices. The remaining subclasses are for P_0^+-matrices and $P_{0,1}^+$-matrices

The study on $wssP_0^+$-matrices is problematic, and it has been explained in Chapter 4 and hence the research on $wssP_{0,1}^+$-matrices: results and discussions are presented in Chapter 4.

Chapter 3

Methodology

This chapter outlines the procedure used to determine whether or not a pattern has weakly sign symmetric $P_{0,1}^{+}$-completion and finally zero completion to a weakly sign symmetric $P_{0,1}^{+}$-matrix.

3.1 Basic concepts

At its basic level, matrix completion problems are like mad libs, the children's word game or sudoku puzzle game. In both situation, you are filling in the blanks, in some cases it is not an easy task. For example, if we consider a sudoku puzzle game. The objective is to fill a 9×9 grid with digits so that each column, each row, and each of the nine 3×3 subgrids that compose the grid contains all of the digits from 1 to 9 (desired property). For matrix completion problems, we also fill in the blanks so that the resulting matrix has certain desired property, and in most cases we look at the entries and determinants. There are many different properties for various classes of matrices but in this research, we have focused on weakly sign symmetric $P_{0,1}^{+}$-matrices.

Graph theory and linear algebra plays an important role in this study on weakly sign symmetric $P_{0,1}^{+}$-completion.

3.2 Completion of patterns

The procedure on how to achieve specific objectives are presented in the following steps:

(i) Present the necessary conditions for weakly sign symmetric $P_{0,1}^{+}$-completion and zero completion to a weakly sign symmetric $P_{0,1}^{+}$-matrix through creations of theorems and some other facts on these completions. This leads to achievement of objective (i).

(ii) Identify various relationships of matrix completions including weakly sign symmetric $P_{0,1}^{+}$-completion and zero completion to weakly sign symmetric $P_{0,1}^{+}$-matrix which can also be presented in form of theorems and corollaries. This leads to achievement of objective (ii).

The classifications of patterns procedure are discussed in the next steps.

(iii) Perform weakly sign symmetric $P_{0,1}^{+}$-completions for digraphs of at most order 4.

(i) Rule out patterns that do not attain necessary conditions presented for in item (i) for weakly sign symmetric $P_{0,1}^{+}$-completions also with the help of relationships with other classes given in item (ii) and the results are presented as lemmas.

(ii) Construct partial weakly sign symmetric $P_{0,1}^{+}$-matrices specifying each of the remaining patterns.

(iii) Compute all principal minors of each partial weakly sign symmetric $P_{0,1}^{+}$-matrices.

(iv) Assign values to all unspecified entries such that it meets the requirement of weakly sign symmetric $P_{0,1}^{+}$-matrices..

(v) Characterize those patterns having weakly sign symmetric $P_{0,1}^{+}$-completion as a theorem. This step leads to achievement of objective (iii).

Since zero completion is stronger than weakly sign symmetric $P_{0,1}^{+}$-completion and we are considering only partial weakly sign symmetric $P_{0,1}^{+}$-matrix, we only investigate those patterns having weakly sign symmetric $P_{0,1}^{+}$-completion in the next step.

(iv) List all patterns having weakly sign symmetric $P_{0,1}^{+}$-completion given in item (iv). Apply zero completion method to those patterns that have been listed; that is, assign all the unspecified entries $x_{ij}s$ to zero such that $\det(\alpha) \geq 0$ for any $\alpha \subseteq \{1, 2, 3, 4\}$ and at least one of each order is positive, and product of twin entries is non-negative then classify those patterns having zero completion to a weakly sign symmetric $P^+_{0,1}$-matrix and is presented a theorem. This will give the results for objective (iv).

These five steps led to achievement of the four objectives of this study.

Chapter 4

Results and discussions

In this chapter we present the main results of this research. Note that throughout the chapter, we denote the entries of a partial matrix A as follows: for $1 \leq i, j \leq n$, the entry a_{ij} denotes a specified non-diagonal entry, the entry x_{ij} an unspecified entry, and the entry d_i denotes a specified diagonal entry (we have studied the situation where all diagonal entries are specified since digraphs are being used in this research). The entry c_{ij} denotes a value assigned to the unspecified entry x_{ij} during the process of completing a partial matrix. A_c is the completion of the partial matrix A.

The main results and discussions are organized as follows:

Section §4.1: Necessary conditions for $wssP_{0,1}^{+}$-completion.

Section §4.2: Relationships between completion problems of $wssP_{0,1}^{+}$-matrix and other class of $wssP$-matrix.

Section §4.3: Classifications of digraphs of at most order 4 having $wssP_{0}^{+}$-completion.

Section §4.4: Classifications of digraphs of at most order 4 having zero completion to a $wssP_{0}^{+}$-matrix.

Although in our analysis, we only considered the situation where all diagonal entries are specified, it is worth discussing the other case briefly.

If a partial $wssP_{0}^{+}$-matrix omits all diagonal entries, then it can be completed to a $wssP_{0}^{+}$-matrix by assigning sufficiently large values to unspecified diagonal entries. A similar argument also applies to a partial $wssP_{0,1}^{+}$-matrix.

The situation changes when some diagonal entries are specified. It is a requirement for the fully specified matrix that one of the diagonal entries is positive, since at least one of the determinants needs to be positive in each order of the matrix.

In general, zeros on the diagonal tend to make completion difficult. Consider the partial $wssP_0^+$-matrix $A = \begin{bmatrix} x & 1 \\ 2 & 0 \end{bmatrix}$ which specifies $Q = \{(1,2),(2,1),(2,2)\}$. This cannot be completed to a $wssP_0^+$-matrix because $\det A_c = -2 < 0$ for any completed matrix A_c of partial matrix A. Thus pattern Q does not have $wssP_0^+$-completion.

As indicated earlier in Chapter 2, the study on weakly sign symmetric P_0^+-completion.

The next result asserts that complete digraphs occur for this class.

Theorem 4.0.1. *The digraphs that have weakly sign symmetric P_0^+-completion are the complete digraphs.*

Proof. Let $wssP_0^+$-matrix A_c be a completion of $wssP_0^+$ matrix A having all diagonal entries specified. Recall that a pattern has $wssP_0^+$-completion if every partial $wssP_0^+$-matrix A can be completed to $wssP_0^+$-matrix A_c. Assume that partial $wssP_0^+$-matrix A has the first $n-1$ diagonal entries as 0 and the last is 1.

Consider the 2×2 principal minor $\det A(i,j)$ for some $i, j \in \{1, \dots, n\}$. Note that $d_i d_j = 0$ always. Now split into three cases:

Case 1: Position ij and ji are specified. In this case we have

$$\det A_c(i,j) = d_i d_j - a_{ij}a_{ji} \geq 0$$

$$so \quad 0 - a_{ij}a_{ji} \geq 0$$

$$so \quad - a_{ij}a_{ji} \geq 0.$$

Thus $a_{ij}a_{ji} \leq 0$ and, by $wssP_0^+$-completion, we have $a_{ij}a_{ji} = 0$.

Case 2: Position ij is specified and ji is unspecified.

$$\det A_c(i,j) = d_i d_j - a_{ij}c_{ji} \geq 0$$
$$so \quad 0 - a_{ij}c_{ji} \geq 0$$
$$so \quad - a_{ij}c_{ji} \geq 0$$

Thus $a_{ij}x_{ji} \leq 0$ and, by $wssP_0^+$-completion, we have $a_{ij}c_{ji} = 0$.

Case 3: Position ij and ji are unspecified.

$$\det A_c(i,j) = d_i d_j - c_{ij}c_{ji} \geq 0$$
$$so \quad 0 - c_{ij}c_{ji} \geq 0$$
$$so \quad - c_{ij}c_{ji} \geq 0$$

Thus $c_{ij}c_{ji} \leq 0$ and, by $wssP_0^+$-completion, we have $c_{ij}c_{ji} = 0$.

Observe that in all cases the product of twin entries is zero. However $wssP_0^+$-completion requires that at least one of the 2×2 principal minors is positive. This is a contradiction. □

This is a new result which shows that the set of digraphs having $wssP_0^+$-completion is a set of complete digraphs.

Most interesting now is that some results in the literature has become trivial. For example, Theorem 4.2.2 in (Sinha, 2015).

From here on, all diagonal entries are assumed to be specified and in particular positive entries since we are dealing with $wssP_{0,1}^+$-matrices.

Before performing any completions, it is always important to study different conditions required for the completion to be achieved, in the next section we present our necessary condition and at some point more than one condition can confirm certain (similar) results on the same digraph.

4.1 Necessary conditions for $wssP_0^+$-completion and $wssP_{0,1}^+$-completion

In this section we have outlined some necessary conditions for $wssP_0^+$-completion and $wssP_{0,1}^+$-completion, and it is under this section that we complete the first objective of this study.

Theorem 4.1.1. *A pattern associated with the null graph of order n has zero completion to a weakly sign symmetric $P_{0,1}^+$-matrix.*

Proof. Consider a pattern $Q = \{(1,1), ..., (n,n)\}$, for any positive integer n. The pattern Q specifies a partial $wssP_{0,1}^+$-matrix A of a null graph. Perform zero completion to A, by assigning all the non-diagonal entries to zeros, that is $x_{ij} = 0$ for all $i \neq j$, gives a $n \times n$ positive diagonal matrix A_c which is a $wssP_{0,1}^+$-matrix, since the principal minor of the principal sub-matrix $A_c(\alpha)$ is $\det A_c(\alpha) > 0$ for all $\alpha \subseteq \{1, ..., n\}$ due to the fact that $\det A(\alpha) = \prod_{i\in\alpha} d_i > 0$. Hence L has a zero completion to a weakly sign symmetric $P_{0,1}^+$-matrix. □

As the consequence of Theorem 4.1.1, the corollary below follows.

Corollary 4.1.2. *A pattern associated with the null graph of order n has a weakly sign symmetric $P_{0,1}^+$-completion.*

The next lemma will help us when showing completions of digraphs using its relationships with sub-digraph having zero completion to a weakly sign symmetric $P_{0,1}^+$-matrix.

Lemma 4.1.3. *Let D be a digraph of order k, and H be an incomplete sub-digraph of order $k-1$. Moreover, assume v to be the unique vertex in $D \setminus H$, and suppose that either the out-degree or the in-degree of v is equal to 0. If H has zero completion to a weakly sign symmetric $P_{0,1}^+$-matrix, then D has zero completion to a weakly sign symmetric $P_{0,1}^+$-matrix.*

Proof. Let digraph D of order k and H an incomplete sub-digraph of order $k-1$ with H having zero completion to a $wssP^{+}_{0,1}$-matrix, then in the process of completing a partial matrix specifying H by assigning zeros to all unspecified entries, we have $\det(\alpha) \geq 0 \quad \forall \alpha \subseteq \{1, \ldots, k-1\}$ and at least one in every order is positive. Note that $\det(1, \ldots, k-1) > 0$ since it is the only one of order $k-1$. Now, if we consider the digraph D of order k with vertex $v \notin H$ i.e., only vertex not in H, and there is no arc pointing away or towards a particular vertex v. The partial matrix specifying D has a column or row corresponding to vertex v with all elements unspecified which can be completed by adding zero, thus giving a positive determinant of D, i.e., $\det(1, \ldots, k) > 0$. Hence D has zero completion to a $wssP^{+}_{0,1}$-matrix. □

Lemma 4.1.3 leads to the theorem below for weakly sign symmetric $P^{+}_{0,1}$-completion, the difference on proof is that the completion of sub-digraphs sometimes does not accept zeros in the next theorem.

Theorem 4.1.4. *Let D be a digraph of order k, and H be an incomplete sub-digraph of order $k-1$. Moreover, assume v to be the unique vertex in $D \setminus H$, and suppose that either the out-degree or the in-degree of v is equal to 0. If H has weakly sign symmetric $P^{+}_{0,1}$-completion, then D has weakly sign symmetric $P^{+}_{0,1}$-completion.*

Proof. The proof follows Lemma 4.1.3, given digraph D of order k and H an incomplete sub-digraph of order $k-1$ with H having $wssP^{+}_{0,1}$-completion, then in the process of completing a partial matrix specifying H not necessarily adding zeros to unspecified entries as in Lemma 4.1.3, we have $\det(\alpha) \geq 0 \quad \forall \alpha \subseteq \{1, \ldots, k-1\}$ and at least one in every order is positive. Note that $\det(1, \ldots, k-1) > 0$ since it is the only one of order $k-1$. Now, if we consider the digraph D of order k with vertex $v \notin H$ i.e., only vertex not in H, and there is no arc pointing away or towards a particular vertex v. The partial matrix specifying D has a column or row corresponding to vertex v with all elements unspecified which can be completed by adding zero, thus giving a positive determinant of D, i.e., $\det(1,\ldots,k) > 0$. Hence D has $wssP^+_{0,1}$-completion.

Recall that zero completion is stronger than weakly sign symmetric $P^+_{0,1}$-completion.

$\square$

Example 4.1.5 below verifies the results of Theorem 4.1.4.

Example 4.1.5.

Let us consider digraph $D_4(4,3)$ and its sub-digraph $D_3(3,4)$ in Figure 4.1.

Figure 4.1: Subdigraph $D_3(3,4)$ and digraph $D_4(4,3)$

An incomplete digraph $D_3(3,4)$ of order 3 has a $wssP^+_{0,1}$-completion then any digraph of order 4 with a vertex not $D_3(3,4)$ of either zero in-degree or out-degree has zero completion to a $wssP^+_{0,1}$-matrix. We consider digraph $D_4(4,3)$, vertex 4 not in $D_3(3,4)$ has zero in-degree and $D_3(3,4)$ is its sub-digraph, so we only need to prove for the $D_3(3,4)$ if it has $wssP^+_{0,1}$-completion. In such a case, by Theorem 4.1.4, digraph $D_4(4,3)$ also has $wssP^+_{0,1}$-completion.

Recall that c_{ij} is a value assigned to unspecified entry x_{ij} and x_{ij} is a specified entry for (i,j)-position.

First, we show that the pattern of $D_3(3,4)$ has $wssP^+_{0,1}$-completion.

The partial $wssP^+_{0,1}$-matrix $A = \begin{bmatrix} d_1 & a_{12} & x_{13} \\ a_{21} & d_2 & x_{23} \\ x_{31} & a_{32} & d_3 \end{bmatrix}$ specifies $D_3(3,4)$.

We consider two cases:

Case 1: $\det A(1,2) > 0$. Then complete the partial matrix by setting all unspecified entries to zero.

Case 2: $\det A(1,2) = 0$. Without loss of generality $a_{12} \neq 0$ and $a_{21} \neq 0$.

In this case we need to show that $\det A(1,3) > 0$ or $\det A(2,3) > 0$, and $\det A > 0$.

Determinants of principal sub-matrices:

$\det A(1,3) = d_1 d_3 - x_{13} x_{31}$; $\quad \det A(2,3) = d_2 d_3 - x_{23} a_{32}$;

$\det A = -a_{32}(d_1 x_{23} - a_{21} x_{13}) + x_{31}(a_{12} x_{23} - d_2 x_{13})$.

Case 2a: If $a_{32} = 0$ then setting $x_{13} = c_{13} = 0$ gives:

$\det A(1,3) = d_1 d_3 > 0$; $\det A(2,3) = d_2 d_3 > 0$; $\det A = a_{12} x_{23} x_{31}$.

Now if $a_{12} > 0$, then set the values of $x_{23} = c_{23}, x_{31} = c_{31} > 0$ or $x_{23} = c_{23}, x_{31} = c_{31} < 0$ and if $a_{12} < 0$, then set the values of $x_{23} = c_{23} < 0$ and $x_{31} = c_{31} > 0$ or $x_{23} = c_{23} > 0$ and $x_{31} = c_{31} < 0$ and this gives:

$\det A(1,3) = d_1 d_3 > 0$; $\det A(2,3) = d_2 d_3 > 0$; $\det A = a_{12} c_{23} c_{31} > 0$.

Case 2b: If $a_{32} \neq 0$ then setting $x_{31} = x_{23} = 0$ gives:

$\det A(1,3) = d_1 d_3 > 0$; $\det A(2,3) = d_2 d_3 > 0$; $\det A = a_{12} a_{32} x_{13}$.

Now if $a_{12} a_{23} > 0$, then set the value $x_{13} = c_{13} > 0$ and if $a_{12} a_{23} < 0$, then set $x_{13} = c_{13} < 0$, this gives:

$\det A(1,3) = d_1 d_3 > 0$; $\det A(2,3) = d_2 d_3 > 0$; $\det A = a_{12} a_{23} c_{13} > 0$.

Thus, in all cases $\det A(1,3) > 0$, $\det A(2,3) > 0$ and $\det A > 0$.

Therefore, any pattern of $D_3(3,4)$ has $wssP^+_{0,1}$-completion.

We have shown that $D_3(3,4)$ has $wssP^+_{0,1}$-completion, and according to Theorem 4.1.4, pattern of $D_4(4,3)$ also has $wssP^+_{0,1}$-completion.

In this example, we have given detail workings instead of just using Theorem

4.1.4.

The partial $wssP^+_{0,1}$-matrix $A = \begin{bmatrix} d_1 & a_{12} & x_{13} & x_{14} \\ a_{21} & d_2 & x_{23} & x_{24} \\ x_{31} & a_{32} & d_3 & x_{34} \\ x_{41} & x_{42} & a_{43} & d_4 \end{bmatrix}$ specifies digraph $D_4(4,3)$.

We consider two cases:

Case 1: $\det A(1,2) > 0$. Then complete the partial matrix by setting all unspecified entries to zero.

Case 2: $\det A(1,2) = 0$. Without loss of generality $a_{12} \neq 0$ and $a_{21} \neq 0$.

In this case we first try to complete $A(1,2,3)$: the partial matrix that specifies $D_3(3,4)$.

In this case we need to show that $\det A(1,3) > 0$ or $\det A(2,3) > 0$, and $\det A > 0$.

Determinants of principal sub-matrices:

$\det A(1,3) = d_1d_3 - x_{13}x_{31}$; $\det A(2,3) = d_2d_3 - x_{23}a_{32}$;

$\det A = -a_{32}(d_1x_{23} - a_{21}x_{13}) + x_{31}(a_{12}x_{23} - d_2x_{13})$.

Case 2a: If $a_{32} = 0$ then setting $x_{13} = c_{13} = 0$ gives:

$\det A(1,3) = d_1d_3 > 0$; $\det A(2,3) = d_2d_3 > 0$; $\det A = a_{12}x_{23}x_{31}$.

Now if $a_{12} > 0$, then set the values of $x_{23} = c_{23}, x_{31} = c_{31} > 0$ or $x_{23} = c_{23}, x_{31} = c_{31} < 0$ and if $a_{12} < 0$, then set the values of $x_{23} = c_{23} < 0$ and $x_{31} = c_{31} > 0$ or $x_{23} = c_{23} > 0$ and $x_{31} = c_{31} < 0$ and this gives:

$\det A(1,3) = d_1d_3 > 0$; $\det A(2,3) = d_2d_3 > 0$; $\det A = a_{12}c_{23}c_{31} > 0$.

Case 2b: If $a_{32} \neq 0$ then setting $x_{31} = x_{23} = 0$ gives:

$\det A(1,3) = d_1d_3 > 0$; $\det A(2,3) = d_2d_3 > 0$; $\det A = a_{12}a_{32}x_{13}$.

Now if $a_{12}a_{23} > 0$, then set the value $x_{13} = c_{13} > 0$ and if $a_{12}a_{23} < 0$, then set $x_{13} = c_{13} < 0$, this gives:

$\det A(1,3) = d_1d_3 > 0$; $\det A(2,3) = d_2d_3 > 0$; $\det A = a_{12}a_{23}c_{13} > 0$.

Thus, in all cases $\det A(1,3) > 0$, $\det A(2,3) > 0$ and $\det A > 0$.

Therefore, pattern $D_3(3,4)$ has $wssP^+_{0,1}$-completion.

In case 2a and 2b, set $x_{14} = x_{24} = x_{34} = x_{41} = x_{42} = 0$ gives $\det A > 0$.

Therefore, any pattern of $D_4(4,3)$ has $wssP^+_{0,1}$-completion.

This also applies to all digraphs of order 4 with a vertex having either zero in-degree or out-degree and not in sub-digraph of order 3 having $wssP^+_{0,1}$-completion.

Example 4.1.5 can not be used to verify Lemma 4.1.3 since digraph $D_3(3,4)$ does not have zero completion to a $wssP^+_{0,1}$-matrix.

The next 3 theorems helps ruling out some patterns having certain properties stated in the Theorems 4.1.6, 4.1.8 and 4.1.9.

Theorem 4.1.6. *Let D be an incomplete digraph of order k, and if there exist two vertices with either in-degree or out-degree equal to $k-1$, then D does not have weakly sign symmetric $P^+_{0,1}$-completion.*

Proof. Consider an incomplete digraph D of order K with two vertices having in-degree equal to $k-1$, then the partial matrices that specifies such digraphs have two of its columns specified. Likewise if they have two vertices having out-degree equal $k-1$, then the partial matrices that specifies such digraphs have two of its rows specified. If we specify entries of the partial matrix A specifying D as follows:

i $d_i = 1$; for $i = 1, \ldots k$.

ii All specified entries of the partial matrix to be 1.

This qualifies to be a $wssP^+_{0,1}$-matrix having either two rows or columns being identical then its determinant is equal to zero. Therefore, D does not have a $wssP^+_{0,1}$-completion. $\square$

Note that if we let A be a square matrix and A^T be its transpose matrix then $\det A = \det A^T$, and therefore, any matrix with identical rows or columns have zero determinant.

Example 4.1.7 below confirms the results of Theorem 4.1.6.

Example 4.1.7.

Figure 4.2: Digraph $D_3(4,4)$ and $D_4(9,8)$

For patterns $D_3(4,3)$ and $D_4(9,8)$ in Figure 4.2, digraph $D_3(4,3)$ of order 3 has 2 vertices 1 and 2 with in-degree equal to 2. Similarly, digraph $D_9(9,8)$ of order 4 has 2 vertices 1 and 2 with out-degree equal to 3 and therefore, according to Theorem 4.1.6, they do not have a $wssP^+_{0,1}$-completion.

We can see it clearly by considering partial $wssP^+_{0,1}$-matrices below:

$$A_1 = \begin{bmatrix} 1 & 1 & x_{13} \\ 1 & 1 & x_{23} \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ x_{41} & x_{42} & x_{43} & 1 \end{bmatrix}$$

specifies $D_3(4,4)$ and $D_4(9,8)$ respectively.

The determinant $\det A_1 = \det A_2 = 0$ for any choice of values for unspecified entries. Hence $D_3(4,3)$ and $D_3(4,4)$ do not have $wssP^+_{0,1}$-completion.

Theorem 4.1.8. *Let D be a digraph of order k and H be an incomplete subdigraph of order $k-1$. Assume also that there exist a vertex v in $D \setminus H$ having* *either in-degree or out-degree equal to* $k-1$. *If* H *does not have a* $wssP^+_{0,1}$-*completion, then* D *does not have a* $wssP^+_{0,1}$-*completion.*

Proof. Let digraph D of order k and an incomplete sub-digraph H of order $k-1$ that does not have $wssP^+_{0,1}$-completion and have a vertex v in $D\backslash H$ with either in-degree or out-degree equal to $k-1$. The partial $k \times k$ $wssP^+_{0,1}$-matrix A specifying D has a partial principal matrix $A(1 \dots k-1)$ specifying H, again the partial matrix A has a row of v (resp. column of v) with all specified entries if in-degree as $k-1$ (resp. if out-degree as $k-1$). In a situation where all specified entries in row or column v are zeros, then $\det A = \det A(1, \dots, k-1)$. If H specified $A(1, \dots, k-1)$ does not have a $wssP^+_{0,1}$-completion then D does not also have a $wssP^+_{0,1}$-completion. $\square$

Example 4.1.7 can be used to elaborate the results of Theorem 4.1.8.

Since it has been shown that digraph $D_3(4,4)$ of order 3 does not have $wssP^+_{0,1}$-completion in Example 4.1.7. Similarly, digraph $D_4(9,8)$ in 4.2 contains $D_3(4,4)$ and vertex 4 which is not in $D_3(4,4)$ with in-degree of 3 hence by the Theorem 4.1.8, $D_4(9,8)$ does not have $wssP^+_{0,1}$-completion. One can also see from the partial matrices A_1 and A_2 in Example 4.1.7, if all specified entries of column 3 has zeros, (as per the theorem) that $\det A_1 = \det A_2 = 0$ and hence both lack $wssP^+_{0,1}$-completion.

It has been stated earlier, immediately before the start of this section that sometimes two or more conditions can give a conclusion on some digraphs.

Theorem 4.1.9. *An incomplete digraph of order* k *does not have zero completion to a* $wssP^+_{0,1}$-*matrix if it has a complete sub-digraph of order* 2.

Proof. Consider a pattern Q associated with a digraph that has a complete sub-digraph of order 2.

Assume that the complete sub-digraph is between vertices 1 and 2. Now let us specify the entries of the partial matrix A as follows:

i $d_i = 1$; for $i = 1, \dots k$.

ii $a_{12} = a_{21} = 1$.

iii All other specified entries are defined to be 0.

Let completed matrix A_c be the zero completion of a partial matrix A. It satisfies $\det A_c = 0$ (because the top rows of A are identical). Therefore, pattern Q does not have zero completion to a $wssP^+_{0,1}$-matrix, as required. □

Example 4.1.10.

Consider a partial $wssP^+_{0,1}$-matrix $A = \begin{bmatrix} d_1 & a_{12} & x_{13} \\ a_{21} & d_2 & a_{23} \\ x_{31} & x_{32} & d_3 \end{bmatrix}$ specifying $D_3(3,1)$. This is an incomplete digraph of order 3 that has a complete sub-digraph of order 2. Assume that $d_1 = d_2 = a_{12} = a_{21} = 1$ and setting all unspecified entries to zero give $\det A(1,2) = \det A = 0$. Hence the pattern does not have zero completion to a $wssP^+_{0,1}$-matrix.

Theorems 4.1.11 and 4.1.12 are very helpful in Section s: p0+ completion.

Theorem 4.1.11. *((DeAlba et al., 2003), Lemma 4.1) All patterns for* 2×2 *matrices have weakly sign symmetric* P*-completion and (weakly) sign symmetric* P_0*-completion. A pattern for* 3×3 *matrices has weakly sign symmetric* P*-completion and (weakly) sign symmetric* P_0*-completion if and only if its digraph does not contain a* 3*-cycle or is complete.*

Theorem 4.1.12. *((DeAlba et al., 2003), Theorem 4.3) Let* Q *be a pattern for* 4×4 *matrices that includes all diagonal positions. The pattern has (weakly) sign* *symmetric P-completion if and only if its digraph is one of the following.*

$q = 0$ $\quad n = 1;$

$q = 1,$ $\quad n = 1;$

$q = 2,$ $\quad n = 1 - 5;$

$q = 3,$ $\quad n = 1 - 11, 13;$

$q = 4,$ $\quad n = 1 - 12, 14 - 19, 21 - 23, 25 - 27;$

$q = 5,$ $\quad n = 1 - 5, 7 - 10, 14 - 17, 21 - 24, 26 - 29, 31, 33 - 34, 36, 37$

$q = 6,$ $\quad n = 1 - 8, 13, 15, 17, 19, 23, 26 - 27, 32, 35, 38 - 40, 43, 46;$

$q = 7,$ $\quad n = 2, 4 - 5, 9, 14, 24, 29, 34, 36;$

$q = 8,$ $\quad n = 1, 10, 12, 18;$

$q = 9,$ $\quad n = 8, 11;$

$q = 12,$ $\quad n = 1.$

4.2 Relationships between completion problems of $wssP^+_{0,1}$-matrix and other class of $wssP$-matrix

In this section we have given relationships between completion problems of $wssP^+_0$-matrix, $wssP^+_{0,1}$-matrix and other class of $wssP$-matrix, and we complete the second objective of this study under this section.

Theorems 4.2.1 to 4.2.10 give the relationships among the sets of patterns having completion of various classes of matrices.

Theorem 4.2.1. *((Sinha, 2015), Corollary 5.2) Any asymmetric pattern that has weakly sign symmetric P-completion also has weakly sign symmetric $P^+_{0,1}$-completion.*

Note that any pattern that has zero completion to a $wssP^+_{0,1}$-matrix also has $wssP^+_{0,1}$-completion. The contrapositive of this statement is that, if a pattern does not have $wssP^+_{0,1}$-completion also does not have zero completion to a $wssP^+_{0,1}$-matrix. This property will help us a great deal in eliminating patterns that lack $wssP^+_{0,1}$-completion when analyzing zero completion to a $wssP^+_{0,1}$-matrix.

Theorem 4.2.2. *((Sinha, 2015), Corollary 4.3) Any pattern that has weakly sign symmetric P^+_0-completion also has weakly sign symmetric $P^+_{0,1}$-completion.*

The contrapositive statement of Theorem 4.2.2: Any pattern that does not have weakly sign symmetric $P^+_{0,1}$-completion does not have weakly sign symmetric P^+_0-completion.

Theorem 4.2.3. *((Sinha, 2015), Corollary 3.1) Any pattern that has weakly sign symmetric $P^+_{0,1}$-completion also has weakly sign symmetric P-completion.*

The contrapositive statement of Theorem 4.2.3: Any pattern that does not have weakly sign symmetric P-completion does not have weakly sign symmetric $P^+_{0,1}$-completion.

From Theorems 4.2.2 and 4.2.3, Corollary 4.2.4 follows.

Corollary 4.2.4. *((Sinha, 2015), Corollary 2.3) Any pattern that has weakly sign symmetric P^+_0-completion also has weakly sign symmetric P-completion.*

The contrapositive statement of Corollary 4.2.4: Any pattern that does not have weakly sign symmetric P-completion does not have weakly sign symmetric P^+_0-completion.

Theorem 4.2.5. *Any pattern that has weakly sign symmetric P^+_0-completion has weakly sign symmetric P_0-completion*

Proof. Let D be a digraph associated to pattern Q having weakly sign symmetric P^+_0-completion and let A be a partial $wwsP^+_0$-matrix specifying D. Clearly, A be a partial $wwsP_0$-matrix. Since every partial $wwsP^+_0$-matrix specifying D is completed to a $wwsP^+_0$-matrix A_c. Again, completed matrix is $wwsP_0$-matrix. Hence digraph D associated to pattern Q has weakly sign symmetric P_0-completion. □

Theorem 4.2.6. *((Hogben, 2003a), Corollary 2.12) Any pattern that has weakly sign symmetric P_0-completion has weakly sign symmetric $P_{0,1}$-completion*

From Theorems 4.2.5 and 4.2.6, Corollary 4.2.7 follows.

Corollary 4.2.7. *Any pattern that has weakly sign symmetric P_0^+-completion also has weakly sign symmetric $P_{0,1}$-completion.*

Theorem 4.2.8. *((Hogben, 2003a), Corollary 2.3) Any pattern that has weakly sign symmetric P_0-completion has weakly sign symmetric P-completion*

Theorem 4.2.9. *Any pattern that has weakly sign symmetric $P_{0,1}^+$-completion also has weakly sign symmetric $P_{0,1}$-completion.*

Proof. The proof technique is similar to the proof of Theorem 4.2.5, we replace P_0^+ in place and $P_{0,1}^+$, in place of P_0 in place of $P_{0,1}$. □

Theorem 4.2.10. *((Hogben, 2003a), Corollary 2.9) Any pattern that has weakly sign symmetric $P_{0,1}$-completion has weakly sign symmetric P-completion*

There are no implications between weakly sign symmetric $P_{0,1}^+$-completion and weakly sign symmetric P_0-completion. This is shown by Example 4.2.11.

Example 4.2.11.

Consider digraphs in Figure 4.3.

Figure 4.3: Digraphs $D_3(4,1)$ and $D_4(4,16)$

Digraphs $D_3(4,1)$ does not contain 3-cycle and according to Theorem 4.1.11, it has weakly sign symmetric P_0-completion.

The partial $wssP^+_{0,1}$-matrix $A = \begin{bmatrix} 1 & 0 & x_{13} \\ 0 & 1 & 1 \\ x_{31} & 1 & 1 \end{bmatrix}$ species $D_3(4,1)$.

In the given partial $wssP^+_{0,1}$-matrix, we need to show that a completed $wssP^+_{0,1}$-matrix satisfies the following:

i det $A(1,3) \geq 0$;

ii det $A > 0$;

iii $x_{13}x_{31} \geq 0$.

First, let us find the determinant of $A(1,3)$.

$$\det A(1,3) = 1 - x_{13}x_{31}.$$

In order to satisfy the condition (i), that is det $A(1,3) \geq 0$, then $x_{13}x_{31} \leq 1$.

Second, let us find the determinant of A.

$$\det A = 1(1-1) - 0(0 - x_{31}) + x_{13}(0 - x_{31}) = -x_{13}x_{31}.$$

Now, in order to satisfy condition (ii), that is det $A > 0$, then $x_{13}x_{31} < 0$.

Condition (iii) says $x_{13}x_{31} \geq 0$ and therefore we have a contradiction. Hence $D_3(4,1)$ do not have a $wssP^+_{0,1}$-completion.

It is now clear that digraph $D_3(4,1)$ have weakly sign symmetric P_0-completion and does not have weakly sign symmetric $P^+_{0,1}$-completion.

Next, we examine digraph $D_4(4,16)$.

According to Theorem 4.1.12, digraph $D_4(4,16)$ have weakly sign symmetric P-completion, since it is asymmetric then by Theorem 4.2.1, digraph $D_4(4,16)$ have weakly sign symmetric $P^+_{0,1}$-completion. Again, by ((DeAlba et al., 2003), Lemma 4.3), it does not have weakly sign symmetric P_0-completion.

Example 4.2.11 shows that there is no relationship between weakly sign symmetric P_0-completion and weakly sign symmetric $P^+_{0,1}$-completion in general.

The relationships between completion problems of $wssP^+_{0,1}$-matrix and other class of $wssP$-matrices can be summarized in Figure 4.4.

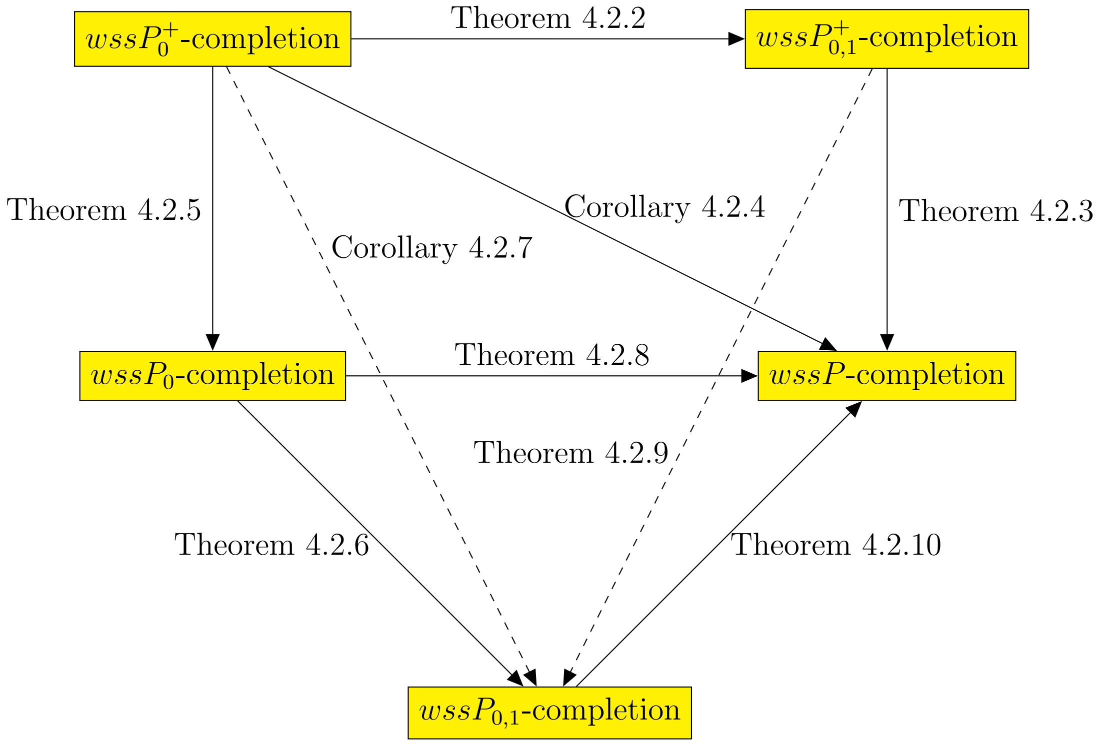

Figure 4.4: Relationship between various classes of $wssP$-matrices

Objectives (iii) is being tackled in Sections 4.3.

The list of all digraphs $D_p(q, n)$ for $1 \leq p \leq 4$ has been given in the Appendix.

The results of Sections 4.1 and 4.2 are used in Sections 4.3 and 4.4.

4.3 Classification of digraphs of order at most 4 having $wssP^+_{0,1}$-completion

In this section we have presented the results of $wssP^+_{0,1}$-completion of patterns associated to digraphs of at most order 4.

Recall that $D_4(q,n)$ where q is number of arcs and n is the diagram number. There are 238 non-isomorphic digraphs of at most order 4 (see, Appendices).

It is important to note first that there is only one (non-isomorphic) digraph of order 1; it is trivial to show that a 1×1 partial matrix has both $wssP^{+}_{0,1}$-completion and zero completion to a $wssP^{+}_{0,1}$-matrix since it specifies complete digraphs.

4.3.1 $wssP^{+}_{0,1}$-completion: digraphs of order 2

In this subsection we will present the results of $wssP^{+}_{0,1}$-completion and zero completion to a $wssP^{+}_{0,1}$-matrix of patterns associated to digraphs of order 2. There are 3 non-isomorphic digraphs of order 3 and are given in the Appendices.

The patterns specifying partial matrices of order 2 are $Q_1 = \{(1,1),(2,2)\}$, $Q_2 = \{(1,1),(1,2),(2,2)\}$ and $Q_3 = \{(1,1),(1,2),(2,1),(2,2)\}$. Patterns Q_1, Q_2 and Q_3 specify $D_2(0,1)$, $D_2(1,1)$ and $D_2(2,1)$ respectively.

Theorem 4.3.1. *: All digraphs of order* 2 *have zero completion to a weakly sign symmetric* $P^{+}_{0,1}$*-matrix.*

Proof. : For all patterns, the product of diagonal entries is positive. In patterns Q_1 and Q_2, we set any unspecified entries to be zero, resulting in the product of non-diagonal entries being zero. Thus, the determinant is positive and the matrix is $wssP^{+}_{0,1}$-matrix, as required.

Finally pattern Q_3 is complete since it specifies a complete digraph. □

Note that zero completion is stronger than plain completion, hence this theorem implies that all digraphs of order 2 have weakly sign symmetric $P^{+}_{0,1}$-completion.

4.3.2 $wssP^{+}_{0,1}$-completion: digraphs of order 3

In this subsection we wish to classify all digraphs of order 3 having $wssP^{+}_{0,1}$-completion. The next lemma and theorem gives our result for $wssP^{+}_{0,1}$-completion for digraphs of order 3. There are 16 non-isomorphic digraphs of order 3 (see, appendices).

Lemma 4.3.2. *If a digraph of order* 3 *is in the following list, then it does not have* $wssP^+_{0,1}$*-completion:*

$$D_3(3,2), D_3(4,1), D_3(4,2), D_3(4,3), D_3(4,4), D_3(5,1)$$

Proof. Each of the digraphs

$$D_3(3,2), D_3(4,2), D_3(5,1)$$

contains a 3-cycle and by Theorem 4.1.11, these digraphs do not have $wssP$-completion. Hence, by Corollary 4.2.4, these digraphs do not have $wssP^+_{0,1}$-completion.

Digraphs $D_3(4,3)$ and $D_3(4,4)$ are of order 3 with 2 vertices having in-degree equal to 2 and out-degree equal to 2 respectively; by Theorem 4.1.6, they do not have a $wssP^+_{0,1}$-completion.

Finally, it has been shown in Example 4.2.11, that digraph $D_3(4,1)$ does not have a $wssP^+_{0,1}$-completion.

□

Theorem 4.3.3. *If a digraph of order* 3 *is in the following list, then it has* $wssP^+_{0,1}$*-completion:*

$$D_3(0,1), D_3(1,1), D_3(2,1), D_3(2,2), D_3(2,3),$$

$$D_3(2,4), D_3(3,1), D_3(3,3), D_3(3,4), D_3(6,1).$$

Proof. Digraph $D_3(0,1)$ is a null digraph and by Corollary 4.1.2, it has $wssP^+_{0,1}$-completion.

Digraph $D_3(6,1)$ is complete since it specifies a complete digraph.

The following of digraphs

$$D_3(1,1), D_3(2,2), D_3(2,3), D_3(2,4), D_3(3,3)$$

does not contain 3-cycle and by Theorem 4.1.11, these digraphs do have $wssP$-completion, again each these digraphs are asymmetric. Thus, by Theorem 4.2.1, these digraphs have $wssP_{0,1}^{+}$-completion.

It has been shown in Example 4.1.5, that digraphs $D_3(3,4)$ have $wssP_{0,1}^{+}$-completion.

Here we analyze each of the remaining 2 digraphs.

First, digraph $D_3(2,1)$.

Consider the partial $wssP_{0,1}^{+}$-matrix $A = \begin{bmatrix} d_1 & a_{12} & x_{13} \\ a_{21} & d_2 & x_{23} \\ x_{31} & x_{32} & d_3 \end{bmatrix}$ specifying $D_3(2,1)$.

We consider two cases:

Case 1: $\det A(1,2) > 0$. Then complete the partial matrix by setting all unspecified entries to zero.

Case 2: $\det A(1,2) = 0$. Without loss of generality $A = \begin{bmatrix} d_1 & d_1 & x_{13} \\ d_2 & d_2 & x_{23} \\ x_{31} & x_{32} & d_3 \end{bmatrix}$

In this case we need to show that $\det A(1,3) > 0$ or $\det A(2,3) > 0$, and $\det A > 0$.

Determinants of principal sub-matrices:

$\det A(1,3) = d_1 d_3 - x_{13}x_{31}$; $\det A(2,3) = d_2 d_3 - x_{23}x_{32}$;

$\det A = -x_{23}(d_1 x_{32} - d_1 x_{31}) + x_{13}(d_2 x_{32} - d_2 x_{31})$.

Setting $x_{31} = c_{31} = 0$ gives:

$\det A(1,3) = d_1 d_3 > 0$; $\det A(2,3) = d_2 d_3 - x_{23}x_{32}$; $\det A = x_{32}(d_2 x_{13} - d_1 x_{23})$.

Setting $x_{23} = c_{23} = 0$ gives:

$\det A(1,3) = d_1 d_3 > 0$; $\det A(2,3) = d_2 d_3 > 0$; $\det A = d_2 x_{13} x_{32}$.

Setting $x_{13} = c_{13} > 0$ and $x_{32} = c_{32} > 0$ gives:

$\det A(1,3) = d_1 d_3 > 0$; $\det A(2,3) = d_2 d_3 > 0$; $\det A = d_2 c_{13} c_{32} > 0$.

Thus $\det A(1,3) > 0$, $\det A(2,3) > 0$ and $\det A > 0$. In both cases the partial $wssP^+_{0,1}$-matrix A is completed to $wssP^+_{0,1}$-matrix A_c Therefore, any pattern of $D_3(2,1)$ has $wssP^+_{0,1}$-completion.

Second, digraph $D_3(3,1)$.

Consider the partial $wssP^+_{0,1}$-matrix $A = \begin{bmatrix} d_1 & a_{12} & x_{13} \\ a_{21} & d_2 & a_{23} \\ x_{31} & x_{32} & d_3 \end{bmatrix}$ specifying $D_3(3,1)$.

We consider two cases:

Case 1: $\det A(1,2) > 0$. Then complete the partial matrix by setting all unspecified entries to zero.

Case 2: $\det A(1,2) = 0$. Without loss of generality $a_{12} \neq 0$ and $a_{21} \neq 0$.

In this case we need to show that $\det A(1,3) > 0$ or $\det A(2,3) > 0$, and $\det A > 0$.

Determinants of principal sub-matrices:

$$\det A(1,3) = d_1 d_3 - x_{13} x_{31},$$

$$\det A(2,3) = d_2 d_3 - a_{23} x_{32},$$

$$\det A = -a_{23}(d_1 x_{32} - a_{12} x_{31}) + x_{13}(a_{21} x_{32} - d_2 x_{31}).$$

Case 2a: If $a_{23} = 0$ then setting $x_{31} = c_{31} = 0$ gives:

$\det A(1,3) = d_1 d_3 > 0$; $\det A(2,3) = d_2 d_3 > 0$; $\det A = a_{21} x_{13} x_{32}$.

Now if $a_{21} > 0$, then set $x_{13} = c_{13}, x_{32} = c_{32} > 0$ or $x_{13} = c_{13}, x_{32} = c_{32} > 0$ and if $a_{21} < 0$, then set $x_{13} = c_{13} > 0$ and $x_{32} = c_{32} < 0$ or $x_{13} = c_{13} < 0$ and $x_{32} = c_{32} > 0$ and this gives gives:

$\det A(1,3) = d_1 d_3 > 0$; $\det A(2,3) = d_2 d_3 > 0$; $\det A = a_{21} c_{13} c_{32} > 0$.

Case 2b: If $a_{23} \neq 0$ then setting $x_{13} = x_{32} = 0$ gives:

$\det A(1,3) = d_1 d_3 > 0$; $\det A(2,3) = d_2 d_3 > 0$; $\det A = a_{12}a_{23}x_{31}$.

Now if $a_{12}a_{23} > 0$, then set the value $x_{31} = c_{31} > 0$ and if $a_{12}a_{23} < 0$, then set $x_{31} = c_{31} < 0$, this gives:

$\det A(1,3) = d_1 d_3 > 0$; $\det A(2,3) = d_2 d_3 > 0$; $\det A = a_{12}a_{23}c_{31} > 0$.

Thus $\det A(1,3) > 0$, $\det A(2,3) > 0$ and $\det A > 0$.

Therefore, any pattern of $D_3(3,1)$ has $wssP^+_{0,1}$-completion.

□

In the next proposition, we give digraphs of order 3 that have $wssP$-completion and do not have $wssP^+_{0,1}$-completion.

Proposition 4.3.4. *If a digraph of order* 3 *is in the following list, then it has $wssP$-completion and does not have $wssP^+_{0,1}$-completion.*

$$D_3(4,1), D_3(4,3), D_3(4,4)$$

Proof. These 3 digraphs do not contain 3-cycle hence by Theorem 4.1.11, they have $wssP$-completion. However, we can observe from the results in Lemma 4.3.2 that they do not have $wssP^+_{0,1}$-completion. □

4.3.3 $wssP^+_{0,1}$-completion: digraphs of order 4

In this subsection we wish to classify all digraphs of order 4 having $wssP^+_{0,1}$-completion. The lemmas and theorem below gives our result for $wssP^+_{0,1}$-completion for digraphs of order 4. There are 218 non-isomorphic digraphs of order 4 (see, Appendices).

Lemma 4.3.5. *If a digraph of order* 4 *is in the following list, then it does not* *have* $wssP_{0,1}^{+}$*-completion:*

$q = 3,\qquad n = 12;$

$q = 4,\qquad n = 13, 20, 24;$

$q = 5,\qquad n = 6, 11-13, 18-20, 25, 30, 32, 35, 38;$

$q = 6,\qquad n = 9-12, 14, 16, 20-22, 24-25, 28-31, 33-34,$

$\qquad 36-37, 41-42, 44-45, 47-48;$

$q = 7,\qquad n = 1, 3, 6-8, 10-13, 15-23, 25-28, 30-33, 35, 37-38;$

$q = 8,\qquad n = 2-9, 11, 13-17, 19-27;$

$q = 9.\qquad n = 1-7, 9-10, 12-13;$

$q = 10,\qquad n = 1-5;$

$q = 11,\qquad n = 1.$

Proof. Each of the digraphs above is not listed in Theorem 4.1.12, therefore, they do not have $wssP$-completion and hence by Theorem 4.2.3 they do not have $wssP_{0,1}^{+}$-completion. □

We will give another set of digraphs that lacks $wssP_{0,1}^{+}$-completion in the next lemma.

Lemma 4.3.6. *If a digraph of order* 4 *is in the following list, then it does not have* $wssP_{0,1}^{+}$*-completion:*

$q = 6;\qquad n = 40, 43;$

$q = 7;\qquad n = 14, 24, 29, 34, 36;$

$q = 8;\qquad n = 10, 12, 18;$

$q = 9;\qquad n = 8, 11.$

Proof. First, digraphs

$$D_4(6,43), D_4(7,36), D_4(8,10), D_4(8,18), D_4(9,11)$$

are of order 4 with 2 vertices having in-degree equal to 3. Again, another set of digraphs

$$D_4(6,40), D_4(7,29), D_4(8,12), D_4(9,8)$$

are of order 4 with 2 vertices having out-degree equal to 3.

By Theorem 4.1.6, all these digraphs do not have a $wssP^+_{0,1}$-completion.

Second, we have already shown in Example 4.2.11 that, digraph $D_3(4,1)$ do not have $wssP^+_{0,1}$-completion.

Digraphs $D_4(7,14)$ and $D_4(7,24)$ are of order 4 and contains $D_3(4,1)$, and also having vertex 4 not in $D_3(4,1)$ of in-degree and out-degree equal to 3 respectively. Again, digraph $D_4(7,34)$ is of order 4 and contains $D_3(4,1)$, and also having vertex 4 not in $D_3(4,1)$ of out-degree equal to 3. Using Theorem 4.1.8, the three digraphs do not have a $wssP^+_{0,1}$-completion. □

Lemma 4.3.7. *Digraphs $D_4(6,2)$, $D_4(7,4)$, $D_4(7,5)$ and $D_4(8,1)$ do not have a $wssP^+_{0,1}$-completion.*

Proof. Let us consider a partial $wssP^+_{0,1}$-matrices $A = \begin{bmatrix} 1 & 1 & 1 & x_{14} \\ 1 & 1 & 1 & x_{24} \\ 1 & 1 & 1 & x_{34} \\ x_{41} & x_{42} & x_{43} & 1 \end{bmatrix}$ specifying $D_4(6,2)$.

For any choice of values for unspecified entries, the determinant $\det A = 0$. Hence $D_4(6,2)$ do not have $wssP^+_{0,1}$-completion.

Digraphs $D_4(7,4)$, $D_4(7,5)$ and $D_4(8,1)$ contains $D_4(6,2)$ as an induced subdigraph. Any partial matrix specifying any of the other 3 digraphs with additional specified entries set to 1 has determinant zero and therefore these digraphs does not have $wssP^+_{0,1}$-completion.

□

The theorem below give digraphs of order 4 that have $wssP^+_{0,1}$-completion.

Theorem 4.3.8. *If a digraph $D_4(q, n)$ is in the following list, then it has $wssP^+_{0,1}$-completion:*

$q = 0,$	$n = 1;$
$q = 1,$	$n = 1;$
$q = 2,$	$n = 1 - 5;$
$q = 3,$	$n = 1 - 11, 13;$
$q = 4,$	$n = 3 - 12, 14 - 19, 21 - 23, 25 - 27;$
$q = 5,$	$n = 8 - 10, 14 - 16, 22 - 24, 26 - 29, 31, 33 - 34, 36 - 37;$
$q = 6,$	$n = 23, 26, 32, 35, 46;$
$q = 12,$	$n = 1.$

Proof. First, we show that $D_4(0, 1)$ and $D_4(12, 1)$ have $wssP^+_{0,1}$-completion.

Digraph $D_4(0, 1)$ is a null digraph and by Corollary 4.1.2, it has $wssP^+_{0,1}$-completion.

Digraph $D_4(12, 1)$ is complete since it specifies a complete digraph.

Second, consider asymmetric digraphs in Theorem 4.1.12, and are listed below.

$q = 1,$	$n = 1;$
$q = 2,$	$n = 2 - 5;$
$q = 3,$	$n = 4 - 11, 13;$
$q = 4,$	$n = 16 - 19, 21 - 23, 25 - 27;$
$q = 5,$	$n = 29, 31, 33 - 34, 36 - 37;$
$q = 6,$	$n = 46.$

All these asymmetric digraphs have $wssP$-completion, and according to Theorem 4.2.1, each of those digraphs in the list above has $wssP^+_{0,1}$-completion.

Third, we will make use of the relationship between digraphs of order 4 and those digraphs of order 3 which have $wssP^+_{0,1}$-completion. Note that digraphs we discuss next are associated to non-asymmetric patterns.

Earlier, we have shown digraphs of order 3 that have $wssP^+_{0,1}$-completion in Theorem 4.3.3. We are interested only in non-asymmetric digraphs of order 3 having $wssP^+_{0,1}$-completion, these are $D_3(2,1)$, $D_3(3,1)$ and $D_3(3,4)$.

For convenience of the readers of this thesis and easy interpretation, in the appendices we have labeled the vertices in a way that incomplete digraphs of order 3 have vertices 1, 2 and 3 and the vertex 4 has either zero in-degree or out-degree for these cases of digraphs. Let us consider the list below:

$$\begin{array}{ll} q = 2, & n = 1; \\ q = 3, & n = 1 - 3; \\ q = 4, & n = 3 - 12; \\ q = 5, & n = 8 - 10, 14 - 16, 22 - 24, 26 - 28 \\ q = 6, & n = 23, 26, 32, 35. \end{array}$$

We will give 3 groups of incomplete digraphs of order 4 containing any of the sub-digraphs $D_3(2,1)$, $D_3(3,1)$ and $D_3(3,4)$ as an induced sub-digraphs and the vertex 4 has either zero in-degree or out-degree.

Group 1: Each of the digraphs

$$D_4(2,1), D_4(3,3), D_4(5,14), D_4(5,24)$$

contains $D_3(2,1)$ and has one of the vertex not in $D_3(2,1)$ having either zero in-degree or out-degree.

Group 2: Each of the digraphs

$$D_4(3,2), D_4(4,4), D_4(4,7), D_4(4,8), D_4(4,11), D_4(4,12), D_4(5,8),$$

$$D_4(5,9), D_4(5,15), D_4(5,16), D_4(5,26), D_4(5,27), D_4(6,26), D_4(6,35)$$

contains $D_3(3,1)$ and has one of the vertex not in $D_3(3,1)$ having either zero in-degree or out-degree.

Group 3: Each of the digraphs

$$D_4(3,1), 4(4,3), D_4(4,5), D_4(4,6), D_4(4,9), D_4(4,10),$$

$$D_4(5,10), D_4(5,22), D_4(5,23), D_4(5,28), D_4(6,23), D_4(6,32)$$

contains $D_3(3,4)$ and has one of the vertex not in $D_3(3,4)$ having either zero in-degree or out-degree.

Now, since each of the digraph in group 1, 2 and 3 contains an incomplete sub-digraph of order 3 that has $wssP^+_{0,1}$-completion, and one of the vertex has either zero in-degree or out-degree, then, by Theorem 4.1.4, all these digraphs have $wssP^+_{0,1}$-completion.

Finally, we will do more analysis for the partial matrices specifying digraphs $D_4(4,14)$ and $D_4(4,15)$. We assign values to all unspecified and try to complete. We will present the results of those digraphs as follows:

First, pattern $D_4(4,14)$

The partial matrix $A = \begin{bmatrix} d_1 & a_{12} & x_{13} & x_{14} \\ a_{21} & d_2 & x_{23} & x_{24} \\ a_{31} & a_{32} & d_3 & x_{34} \\ x_{41} & x_{42} & x_{43} & d_4 \end{bmatrix}$ specifies digraph $D_4(4,14)$

We will consider two cases:

Case 1: $\det A(1,2) > 0$. Then complete the partial matrix by setting all unspecified entries with zeros.

Case 2: $\det A(1,2) = 0$. Without loss of generality $a_{12} \neq 0$ and $a_{21} \neq 0$. Then set $x_{34} = c_{34}$ and $x_{43} = c_{43}$ with conditions that $c_{43} > 0$ and $0 < c_{34} < d_3 d_4 c_{43}^{-1}$, and all other unspecified entries to zero.

Now, the partial matrix can be completed to $A_c = \begin{bmatrix} d_1 & a_{12} & 0 & 0 \\ a_{21} & d_2 & 0 & 0 \\ a_{31} & a_{32} & d_3 & c_{34} \\ 0 & 0 & c_{43} & d_4 \end{bmatrix}$.

This can be seen by looking at the principal minors:

$\det A(1,2) = 0$; $\det A(1,3) = d_1 d_3 > 0$; $\det A(1,4) = d_1 d_4 > 0$;

$\det A(2,3) = d_2 d_3 > 0$; $\det A(2,4) = d_2 d_4 > 0$; $\det A(3,4) = d_3 d_4 - c_{34} c_{43} > 0$;

$\det A(1,2,3) = 0$; $\det A(1,2,4) = 0$; $\det A(1,3,4) = d_1(d_3 d_4 - c_{34} c_{43}) > 0$;

$\det A(2,3,4) = d_2(d_3 d_4 - c_{34} c_{43}) > 0$; $\det A = a_{12} a_{21} c_{34} c_{43} > 0$.

Since all determinants are nonnegative and at least one in every order is positive, then the partial weakly sign symmetric $P_{0,1}^{+}$-matrix has been completed to a weakly sign symmetric $P_{0,1}^{+}$-matrix.

Therefore, any pattern of $D_4(4,14)$ has $wssP_{0,1}^{+}$-completion.

Second, pattern $D_4(4,15)$

The partial matrix $A = \begin{bmatrix} d_1 & a_{12} & a_{13} & x_{14} \\ x_{21} & d_2 & x_{23} & x_{24} \\ a_{31} & a_{32} & d_3 & x_{34} \\ x_{41} & x_{42} & x_{43} & d_4 \end{bmatrix}$ specifies digraph $D_4(4,15)$.

We will consider two cases:

Case 1: $\det A(1,3) > 0$. Then complete the partial matrix by setting all unspecified entries with zeros.

Case 2: $\det A(1,3) = 0$. Without loss of generality $a_{13} \neq 0$ and $a_{31} \neq 0$. Then set $x_{24} = c_{24}$ and $x_{42} = c_{42}$ with conditions that $c_{42} > 0$ and $0 < c_{24} < d_3 d_4 c_{42}^{-1}$, and all other unspecified entries to zero.

Now, the partial matrix can be completed to $A_c = \begin{bmatrix} d_1 & a_{12} & a_{13} & 0 \\ 0 & d_2 & 0 & c_{24} \\ a_{31} & a_{32} & d_3 & 0 \\ 0 & c_{42} & 0 & d_4 \end{bmatrix}$.

This can be seen by looking at the principal minors:

$\det A(1,2) > 0$; $\det A(1,3) = 0$; $\det A(1,4) = d_1 d_4 > 0$;

$\det A(2,3) = d_2 d_3 > 0$; $\det A(2,4) = d_2 d_4 - c_{24}c_{42} > 0$;

$\det A(3,4) = d_3 d_4 > 0$; $\det A(1,2,3) = 0$;

$\det A(1,2,4) = d_1(d_2 d_4 - x_{24}x_{42}) > 0$; $\det A(1,3,4) = 0$;

$\det A(2,3,4) = d_4(d_2 d_4 - c_{24}c_{42}) > 0$; $\det A = a_{13}a_{31}c_{24}c_{42} > 0$.

Since all determinants are nonnegative and at least one in every order is positive, then the partial weakly sign symmetric $P_{0,1}^{+}$-matrix has been completed to weakly sign symmetric $P_{0,1}^{+}$-matrix.

Therefore, any pattern of $D_4(4,15)$ has $wssP_{0,1}^{+}$-completion.

□

In the next proposition, we give digraphs of order 4 that have $wssP$-completion and do not have $wssP_{0,1}^{+}$-completion.

Proposition 4.3.9. *If a digraph $D_4(q,n)$ is in the following list, then it has $wssP$-completion and does not have $wssP_{0,1}^{+}$-completion:*

$q = 6,$	$n = 2, 40, 43;$
$q = 7,$	$n = 4, 5, 14, 24, 29, 34, 36;$
$q = 8,$	$n = 1, 10, 12, 18;$
$q = 9,$	$n = 8, 11.$

Proof. By Theorem 4.1.12, these digraphs have $wssP$-completion. However, we can observe from the results in Lemma 4.3.6 and 4.3.7 that they do not have $wssP_{0,1}^{+}$-completion. □

When we eliminate digraphs in Lemma 4.3.5, 4.3.6 and 4.3.7 which do not have $wssP_{0,1}^{+}$-completion and those digraphs in Theorem 4.3.8 which have $wssP_{0,1}^{+}$-completion, we remain with digraphs which have $wssP$-completion and we do not know if they have $wssP_{0,1}^{+}$-completion, leaving our classification under this subsection incomplete. The list is given below.

$q = 4, \quad n = 1 - 2;$

$q = 5, \quad n = 1 - 5, 7, 17, 21;$

$q = 6, \quad n = 1, 3 - 8, 13, 15, 17, 19, 27, 38 - 39;$

$q = 7, \quad n = 2, 9.$

The next section gives the results for fourth objective of this study.

4.4 Classification of digraphs of order at most 4 having zero completion to a $wssP_{0,1}^{+}$-matrix

In this section we have presented the results of zero completion to a $wssP_{0,1}^{+}$-matrix of patterns associated to digraphs of at most order 4. Our main results are in form of theorems, which gives a complete classification.

We have already shown in Theorem 4.3.1, that all digraphs of order 2 have zero completion to a weakly sign symmetric $P_{0,1}^{+}$-matrix.

In Theorem 4.4.1, we wish to classify all digraphs of order 3 for which we have zero completion to a $wssP_{0,1}^{+}$-matrix.

Theorem 4.4.1. *A digraph of order* 3 *has zero completion to a* $wssP_{0,1}^{+}$*-matrix if it lies in the following list:*

$$D_3(0,1), D_3(1,1), D_3(2,2), D_3(2,3), D_3(2,4), D_3(3,3), D_3(6,1).$$

Proof. Zero completion is stronger than $wssP_{0,1}^{+}$-completion, and therefore we will only consider digraphs listed in Theorem 4.3.3.

According to Theorem 4.1.9, incomplete digraphs with a complete sub-digraph of order 2 don't have zero completion to a $wssP_{0,1}^{+}$-matrix and we therefore rule out those digraphs from the list in Theorem 4.3.3.

Now, the remaining digraphs are asymmetric digraphs of order 3 having $wssP_{0,1}^{+}$-completion, $D_3(0,1)$ and $D_3(6,1)$, and they are discussed as follows:

Digraph $D_3(0,1)$ is a null digraph and by Theorem 4.1.1, it has zero completion to a $wssP_{0,1}^{+}$-matrix.

Digraph $D_3(6,1)$ is complete since it specifies a complete digraph.

For $D_3(1,1)$, $D_3(2,2)$, $D_3(2,3)$ and $D_3(2,4)$, we will relate with digraph $D_2(1,1)$. We have earlier shown in Theorem 4.3.1, that $D_2(1,1)$ has zero completion to a $wssP_{0,1}^{+}$-matrix.

Digraphs $D_3(1,1)$, $D_3(2,2)$ and $D_3(2,3)$ of order 3 contains sub-digraph $D_2(1,1)$ as of order 2 (with vertex 1 and 2) and vertex 3 has a zero out-degree, then by Lemma 4.1.3, they have zero completion to a weakly sign symmetric $P_{0,1}^{+}$-matrix.

Digraph $D_3(2,4)$ of order 3 contain sub-digraph $D_2(1,1)$ of order 2 (with vertex 1 and 2) and vertex 3 has a zero in-degree, then again by Lemma 4.1.3, they have zero completion to a weakly sign symmetric $P_{0,1}^{+}$-matrix.

The only remaining pattern is $D_3(3,3)$.

The partial $wssP_{0,1}^{+}$-matrix $A = \begin{bmatrix} d_1 & a_{12} & a_{13} \\ x_{21} & d_2 & x_{23} \\ x_{31} & a_{32} & d_3 \end{bmatrix}$ specifies $D_3(3,3)$.

Determinants of Principal Submatrices

$$\det A(1,2) = d_1 d_2 - a_{12}x_{21}; \qquad \det A(1,3) = d_1 d_3 - a_{13}x_{31};$$

$$\det A(2,3) = d_2 d_3 - x_{23}a_{32};$$

$$\det A = d_1(d_2 d_3 - x_{23}a_{32}) - a_{12}(d_3 x_{21} - x_{23}x_{31}) + a_{13}(x_{21}a_{32} - d_2 x_{31}).$$

Perform zero completion by setting all unspecified entries of A to zeros gives:

$$\det A(1,2) = d_1 d_2 > 0; \qquad \det A(1,3) = d_1 d_3 > 0;$$

$$\det A(2,3) = d_2 d_3 > 0; \qquad \det A = d_1 d_2 d_3 > 0.$$

Since all the principal minors are positive then digraph $D_3(3,3)$ has zero completion to a $wssP^+_{0,1}$-matrix. □

By examining the digraphs given in the list in Theorem 4.4.1, we obtain the following corollary:

Corollary 4.4.2. *A digraph of order 3 has zero completion to a $wssP^+_{0,1}$-matrix if and only if it is null, complete or is asymmetric and does not contain a 3-cycle.*

In Theorem 4.4.3, we wish to classify digraphs of order 4 which have zero completion to a $wssP^+_{0,1}$-matrix.

Theorem 4.4.3. *A digraph of order 4 has zero completion to a $wssP^+_{0,1}$-matrix if and only if it lies in the following list:*

$$
\begin{array}{ll}
q = 0; & n = 1; \\
q = 1; & n = 1; \\
q = 2; & n = 2 - 5; \\
q = 3; & n = 4 - 11, 13; \\
q = 4; & n = 16 - 19, 21 - 23, 25 - 27; \\
q = 5; & n = 29, 31, 33 - 34, 36 - 37; \\
q = 6; & n = 46; \\
q = 12 & n = 1.
\end{array}
$$

Proof.

Digraph $D_4(0,1)$ is a null digraph and by Theorem 4.1.1, it has zero completion to a $wssP^+_{0,1}$-matrix.

Digraph $D_4(12,1)$ is complete since it specifies a complete digraph.

We will first rule out those incomplete digraphs that have a complete digraph of order 2 since by Theorem 4.1.9, they don't have zero completion to a $wssP^+_{0,1}$-matrix. Again, zero completion is strong than a $wssP^+_{0,1}$-completion and therefore we will discuss patterns that do have $wssP^+_{0,1}$-completion. From these two conditions, it is now clears that, the only patterns to consider are asymmetric patterns that have $wssP^+_{0,1}$-completion, and are given in the list below.

$q = 1;$	
$q = 2;$	$n = 2 - 5;$
$q = 3;$	$n = 4 - 11, 13;$
$q = 4;$	$n = 16 - 19, 21 - 23, 25 - 27;$
$q = 5;$	$n = 29, 31, 33 - 34, 36 - 37$
$q = 6;$	$n = 46.$

In order to find out if these digraphs have zero completion to a $wssP^+_{0,1}$-matrix, it will be more easy to relate with digraphs of order 3 than computing their principal minors and assigning zero to all unspecified entries. We have earlier shown in Theorem 4.4.1 incomplete digraphs of order 3 that have zero completion to a $wssP^+_{0,1}$-matrix and we will make use of the digraphs $D_3(1,1)$, $D_3(2,2)$, $D_3(2,3)$, $D_3(2,4)$ and $D_3(3,3)$ in this subsection.

We will give 5 groups of incomplete digraphs of order 4 that containing any of the sub-digraphs above as an induced sub-digraphs and has vertex 4 not in the sub-digraphs having either zero in-degree or out-degree.

Group 1: Each of the digraphs

$$D_4(1,1), D_4(2,5)$$

contains $D_3(1,1)$ and has one of the vertex not in $D_3(1,1)$ having either zero in-degree or out-degree.

Group 2: Each of the digraphs

$$D_4(2,2), D_4(3,4), D_4(3,9), D_4(3,10), D_4(4,17)$$

contains $D_3(2,2)$ and has one of the vertex not in $D_3(2,2)$ having either zero in-degree or out-degree.

Group 3: Each of the digraphs

$$D_4(2,4), 4(3,7), D_4(3,8), D_4(4,19), D_4(5,34)$$

contains $D_3(2,3)$ and has one of the vertex not in $D_3(2,3)$ having either zero in-degree or out-degree.

Group 4: Each of the digraphs

$$D_4(2,3), 4(3,5), D_4(3,6), D_4(3,11), D_4(4,18)$$

contains $D_3(2,4)$ and has one of the vertex not in $D_3(2,4)$ having either zero in-degree or out-degree.

Group 5: Each of the digraphs

$$D_4(3,13), D_4(4,21), D_4(4,22), D_4(4,23), D_4(4,25), D_4(4,26),$$

$$D_4(4,27), D_4(5,29), D_4(5,31), D_4(5,33), D_4(5,36), D_4(5,37), D_4(6,46)$$

contains $D_3(3,3)$ and has one of the vertex not in $D_3(3,3)$ having either zero in-degree or out-degree.

Now, since each of the digraph in group 1, 2, 3, 4 and 5 contains a incomplete sub-digraph of order 3 that has zero completion to a $wssP^+_{0,1}$-matrix, and one of the vertex not in the sub-digraph has either zero in-degree or out-degree, then, by Lemma 4.1.3, all these digraphs have zero completion to a $wssP^+_{0,1}$-matrix.

The only remaining asymmetric digraph that has $wssP^+_{0,1}$-completion is $D_4(4,16)$, we will give the analysis as follows:

We will first assign zeros to all unspecified entries of a partial matrix specifying $D_4(4,16)$, then compute the principal minors and see if it meets the conditions of being a $wssP^+_{0,1}$-matrix.

Consider a completed $wssP^+_{0,1}$-matrix $A_c = \begin{bmatrix} d_1 & a_{12} & 0 & 0 \\ 0 & d_2 & a_{23} & 0 \\ 0 & 0 & d_3 & a_{34} \\ a_{41} & 0 & 0 & d_4 \end{bmatrix}$ of $D_4(4,16)$.

The principal minors of the completed $wssP^+_{0,1}$-matrix A_c.

$\det A(1,2) = d_1d_2 > 0$; $\det A(1,3) = d_1d_3 > 0$; $\det A(1,4) = d_1d_4 > 0$;

$\det A(2,3) = d_2d_3 > 0$; $\det A(2,4) = d_2d_4 > 0$; $\det A(3,4) = d_3d_4 > 0$;

$\det A(1,2,3) = d_1d_2d_3 > 0$; $\det A(1,2,4) = d_1d_2d_4 > 0$;

$\det A(1,3,4) = d_1d_3d_4 > 0$; $\det A(2,3,4) = d_2d_3d_4 > 0$; $\det A > 0$.

All the determinants are positive, then every partial matrix A of $D_4(4,16)$ can be completed to a $wssP^+_{0,1}$-matrix by assigning all unspecified entries to zero, hence the pattern $D_4(4,16)$ has zero completion to a $wssP^+_{0,1}$-matrix.

$\square$

By examining the digraphs given in the list in Theorem **??**, we obtain the following corollary:

Corollary 4.4.4. *Any asymmetric digraph of order 4 having $wssP^+_{0,1}$-completion also has zero completion to a $wssP^+_{0,1}$-matrix.*

Chapter 5

Summary, conclusion and recommendation

This chapter presents a summary of the key results of the study based on the research objectives and the conclusions drawn out of these results. The chapter also presents the recommendations of the study and suggests areas revealed by the study that require further investigations.

5.1 Summary

The aim of the study was to establish classifications of digraphs specified by $wssP_{0,1}^{+}$-matrices. Recall that digraphs are denoted as $D_p(q,n)$ where p, q and n denotes number of vertices, number of arcs and the diagram number respectively.

Based on the results of Chapter 4, we have given our classifications for digraphs up to order 4 as follows:

1. For $1 \leq p \leq 4$, the digraph $D_p(q,n)$ has zero completion to a $wssP_{0,1}^{+}$-matrix if and only if it lies in the following list:

$$
\begin{array}{lll}
p = 1; & q = 0, & n = 1; \\
p = 2; & q = 0, & n = 1; \\
 & q = 1, & n = 1; \\
 & q = 2, & n = 1;
\end{array}
$$

$p = 3;$	$q = 0,$	$n = 1;$
	$q = 1,$	$n = 1;$
	$q = 2,$	$n = 2 - 4;$
	$q = 3,$	$n = 3;$
	$q = 6,$	$n = 1;$
$p = 4;$	$q = 0,$	$n = 1;$
	$q = 1,$	$n = 1;$
	$q = 2,$	$n = 2 - 5;$
	$q = 3,$	$n = 4 - 11, 13;$
	$q = 4,$	$n = 16 - 19, 21 - 23, 25 - 27.$
	$q = 5,$	$n = 29, 31, 33 - 34, 36 - 37;$
	$q = 6,$	$n = 46;$
	$q = 12,$	$n = 1.$

2. If a digraph $D_p(q, n)$ is in the following list, then it has $wssP^+_{0,1}$-completion:

$p = 1$		
$p = 2;$	$q = 0,$	$n = 1;$
	$q = 1,$	$n = 1;$
	$q = 2,$	$n = 1;$
$p = 3;$	$q = 0,$	$n = 1;$
	$q = 1,$	$n = 1;$
	$q = 2,$	$n = 1 - 4;$
	$q = 3,$	$n = 1, 3 - 4;$
	$q = 6,$	$n = 1;$

$p = 4;$ $q = 0,$ $n = 1;$

$q = 1,$ $n = 1;$

$q = 2,$ $n = 1 - 5;$

$q = 3,$ $n = 1 - 11, 13;$

$q = 4,$ $n = 3 - 12, 14 - 19, 21 - 23, 25 - 27;$

$q = 5,$ $n = 8 - 10, 14 - 16, 22 - 24, 26 - 29, 31, 33 - 34, 36 - 37;$

$q = 6,$ $n = 23, 26, 32, 35, 46;$

$q = 12,$ $n = 1.$

3. If a digraph $D_p(q, n)$ is in the following list, then it does not have $wssP^+_{0,1}$-completion:

$p = 3;$ $q = 3,$ $n = 2;$

$q = 4,$ $n = 1 - 4;$

$q = 5,$ $n = 1;$

$p = 4;$ $q = 3,$ $n = 12;$

$q = 4,$ $n = 13, 20, 24;$

$q = 5,$ $n = 6, 11 - 13, 18 - 20, 25, 30, 32, 35, 38;$

$q = 6,$ $n = 2, 9 - 12, 14, 16, 18, 20 - 22, 24 - 25, 28 - 31,$

$33 - 34, 36 - 37, 40 - 43, 44 - 45, 47 - 48;$

$q = 7,$ $n = 1, 3 - 8, 10 - 38;$

$q = 8,$ $n = 1 - 27;$

$q = 9,$ $n = 1 - 13;$

$q = 10,$ $n = 1 - 5;$

$q = 11,$ $n = 1.$

4. If a digraph $D_p(q,n)$ is in the following list, then it has $wssP$-completion and does not have $wssP^+_{0,1}$-completion:

$p=3;$	$q=4,$	$n=1,3,4;$
$p=4;$	$q=6,$	$n=2,40,43;$
	$q=7,$	$n=4,5,14,24,29,34,36;$
	$q=8,$	$n=1,10,12,18;$
	$q=9,$	$n=8,11.$

5. If a digraphs is in the following list, then it has $wssP$-completion and not known whether or not it has $wssP^+_{0,1}$-completion:

$p=4;$	$q=4,$	$n=1-2;$
	$q=5,$	$n=1-5,7,17,21;$
	$q=6,$	$n=1,3-8,13,15,17,19,27,38-39;$
	$q=7,$	$n=2,9.$

5.2 Conclusion

The following conclusions were made in relation to the objectives of this study. Figure 5.1 shows relationships among the sets of patterns having different completions for partial $wssP^+_{0,1}$-matrices. Our conclusions will be summarized by the explanations of Figure 5.1.

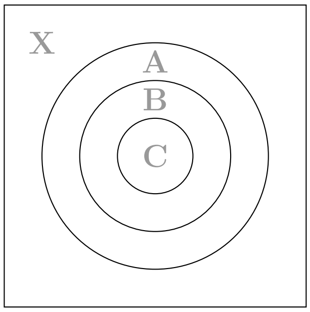

Figure 5.1: Relationships among the sets of patterns having completions for $wssP_{0,1}^{+}$-matrices and $wssP$-matrices.

Key

X: Number of patterns considered.

A: Patterns that have $wssP$-completion.

B: Patterns that have $wssP_{0,1}^{+}$-completion.

C: Patterns that have zero completion to a $wssP_{0,1}^{+}$-matrix.

Set **X** gives the venn space or the boundary for number of digraphs to consider in a study, most matrix completions considers up to order 4 digraphs which is a similar case in this study. We have considered digraphs of order 1, 2, 3, and 4 with 1, 3, 16 and 218 non-isomorphic digraphs respectively; they are 238 digraphs in total. The venn space expands depending on the order of digraphs considered in a study. Set **X** has 114 digraphs which are not in set A, those are the set of digraphs in complement set A^C, that is, $|A^c| = 114$.

The next sets changes depending on the type of completions and their relationships for digraphs are presented as follows:

First, set **A** is the set of patterns that have $wssP$-completion; this the superset of all the patterns and it was used as restriction for the number of patterns to consider in this study. The cardinality of this set is 124, that is $|A| = 124$.

Second, set **B** is the set of patterns that have $wssP_{0,1}^{+}$-completion; this is the biggest set in this study for which we did not get complete classifications for all digraphs of up to order 4 and after partial classifications it was found that the cardinality of this set is between 79 and 105, that is $79 \leq |B| \leq 105$.

Third, set **C** is the set of patterns that have zero completion to a $wssP^+_{0,1}$-matrix; this is a special set of **B** which we have successfully classified all digraphs of up to order 4 and the cardinality of this set is 44, that is $|C| = 44$. These are asymmetric patterns associated with digraphs of at most order 4 having weakly sign symmetric P-completion.

5.3 Recommendation

Most matrix completion problem research work considers digraphs of up to order 4, in this study we have given complete classifications of digraphs of at most order 4 having zero completion to a $P^+_{0,1}$-matrix. On the classifications of digraphs having $wssP^+_{0,1}$-completion, we have also considered digraphs of up to order 4 and we were not able to give complete classifications, and we, therefore, recommend for a study on the determination of stronger necessary and sufficient conditions for weakly sign symmetric $P^+_{0,1}$-completion to classify the remaining 26 digraphs listed below.

$$
\begin{array}{lll}
p = 4; & q = 4, & n = 1 - 2; \\
 & q = 5, & n = 1 - 5, 7, 17, 21; \\
 & q = 6, & n = 1, 3 - 8, 13, 15, 17, 19, 27, 38 - 39; \\
 & q = 7, & n = 2, 9.
\end{array}
$$

5.4 Suggestions for further research

The study suggests for similar studies to be conducted on the following classes:

(i) Sign symmetric P_0-matrices and positive P_0-matrices.

(ii) Nonnegative $P_{0,1}$-matrices and positive $P_{0,1}$-matrices.

(iii) Sign symmetric P^+_0-matrices, nonnegative P^+_0-matrices and positive P^+_0-matrices.

(iv) Sign symmetric $P_{0,1}^{+}$-matrices, nonnegative $P_{0,1}^{+}$-matrices and positive $P_{0,1}^{+}$-matrices.

Again we suggest further research on the unclassified patterns for P-matrices and $P_{0,1}^{+}$-matrices classes.

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Appendices

Appendix: Digraphs of at most order 4

We have provided all the non-isomorphic digraphs of order 2, 3 and 4 as given in (Harary, 1969). This additional information will help the reader in knowing every digraph used in this study. These digraphs are denoted as $D_p(q, n)$ where p denotes the number of vertices, q denotes the number of arcs and n denotes the diagram number for digraphs having same number of vertices and arcs. The serial number n is important in distinguishing non-isomorphic digraphs having the same number of vertices and arcs.

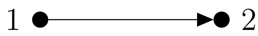
The table below shows digraphs of up to order 5 and respective number of non-isomorphic digraphs.

Table 1: Digraphs of at most order 4

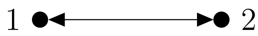
Number of vertice(s) p	Number of arcs(s) q	Non-isomorphic digraphs having q arcs(s)	Non-isomorphic digraphs having p vertices(s)
1	0	1	1
2	0 1 2	1 1 1	3
3	0 1 2 3 4 5 6	1 1 4 4 4 1 1	16
4	0 1 2 3 4 5 6 7 8 9 10 11 12	1 1 5 13 27 38 48 38 27 13 5 1 1	218
5			9608

Digraphs of order 2

$D_2(0,1)$

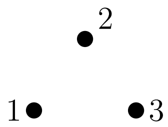


$D_2(1,1)$

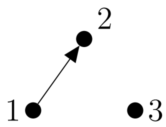


$D_2(2,1)$

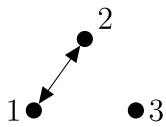
Digraphs of order 3



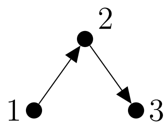
$D_3(0,1)$



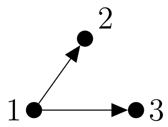
$D_3(1,1)$



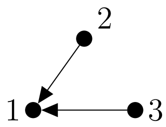
$D_3(2,1)$



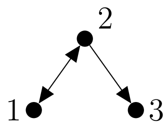
$D_3(2,2)$



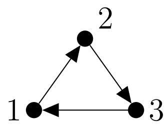
$D_3(2,3)$



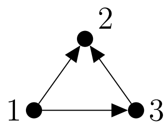
$D_3(2,4)$



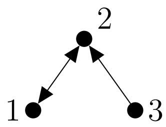
$D_3(3,1)$



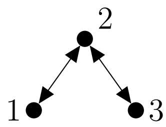
$D_3(3,2)$



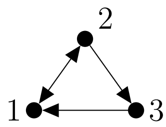
$D_3(3,3)$



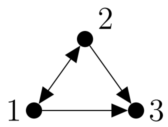
$D_3(3,4)$



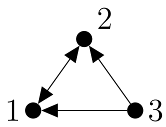
$D_3(4,1)$



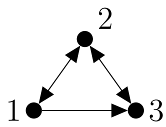
$D_3(4,2)$



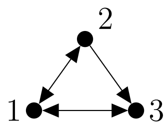
$D_3(4,3)$



$D_3(4,4)$

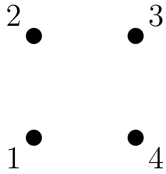


$D_3(5,1)$

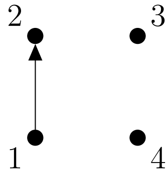


$D_3(6,1)$

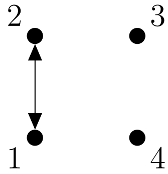
Digraphs of order 4 and at most 3 arcs



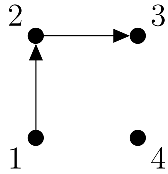
$D_4(0,1)$



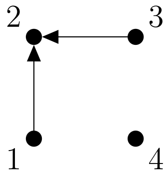
$D_4(1,1)$



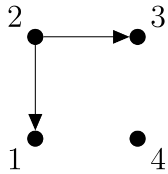
$D_4(2,1)$



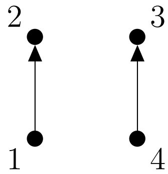
$D_4(2,2)$



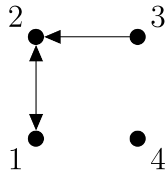
$D_4(2,3)$



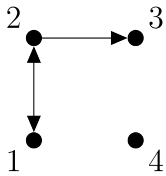
$D_4(2,4)$



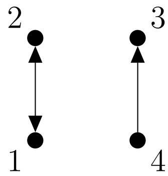
$D_4(2,5)$



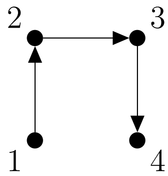
$D_4(3,1)$



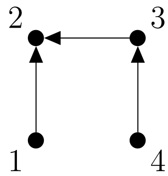
$D_4(3,2)$



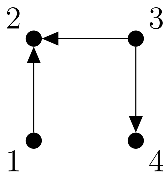
$D_4(3,3)$



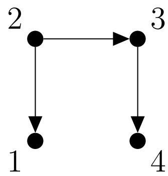
$D_4(3,4)$



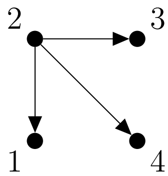
$D_4(3,5)$



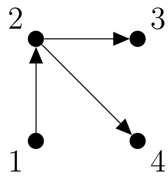
$D_4(3,6)$



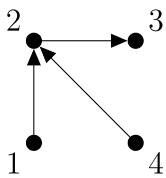
$D_4(3,7)$



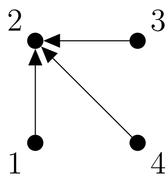
$D_4(3,8)$



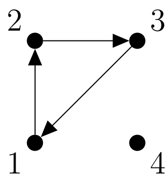
$D_4(3,9)$



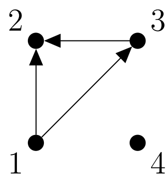
$D_4(3,10)$



$D_4(3,11)$

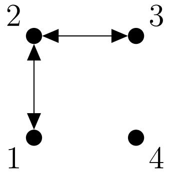


$D_4(3,12)$

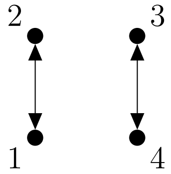


$D_4(3,13)$

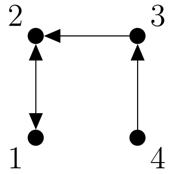
Digraphs of order 4 and 4 arcs



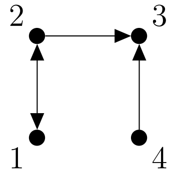
$D_4(4,1)$



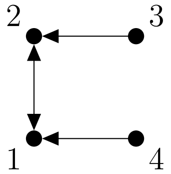
$D_4(4,2)$



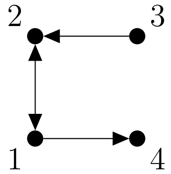
$D_4(4,3)$



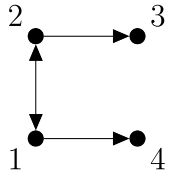
$D_4(4,4)$



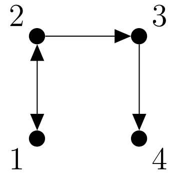
$D_4(4,5)$



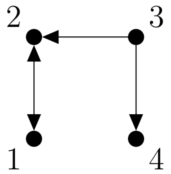
$D_4(4,6)$



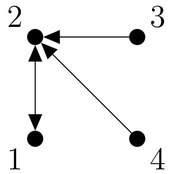
$D_4(4,7)$



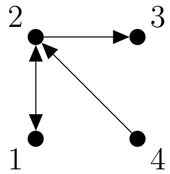
$D_4(4,8)$



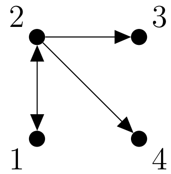
$D_4(4,9)$



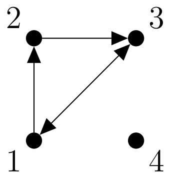
$D_4(4,10)$



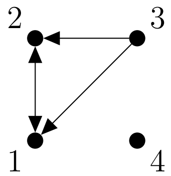
$D_4(4,11)$



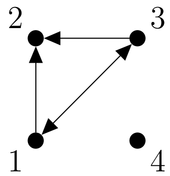
$D_4(4,12)$



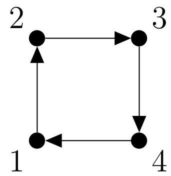
$D_4(4,13)$



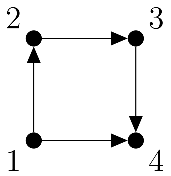
$D_4(4,14)$



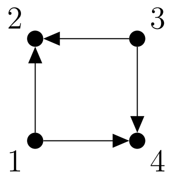
$D_4(4,15)$



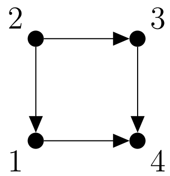
$D_4(4,16)$



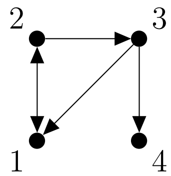
$D_4(4,17)$



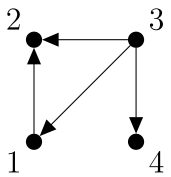
$D_4(4,18)$



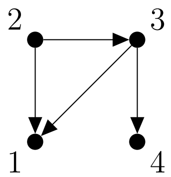
$D_4(4,19)$



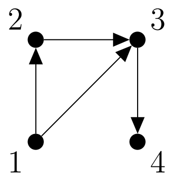
$D_4(4,20)$



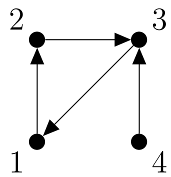
$D_4(4,21)$



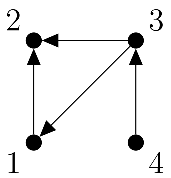
$D_4(4,22)$



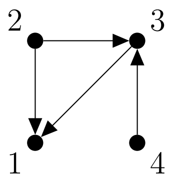
$D_4(4,23)$



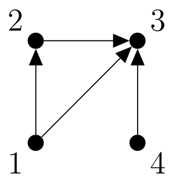
$D_4(4,24)$



$D_4(4,25)$

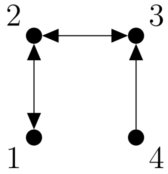


$D_4(4,26)$

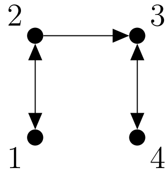


$D_4(4,27)$

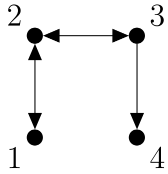
Digraphs of order 4 and 5 arcs: $D_4(5, 1-28)$



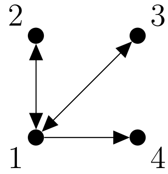
$D_4(5,1)$



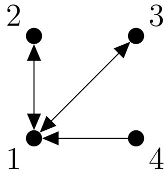
$D_4(5,2)$



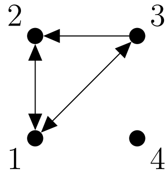
$D_4(5,3)$



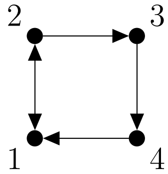
$D_4(5,4)$



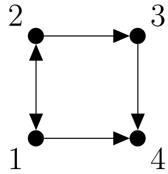
$D_4(5,5)$



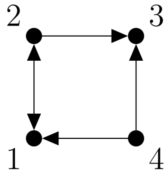
$D_4(5,6)$



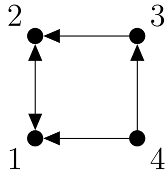
$D_4(5,7)$



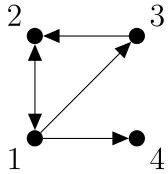
$D_4(5,8)$



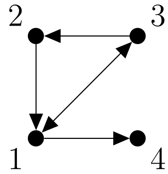
$D_4(5,9)$



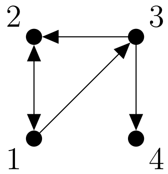
$D_4(5,10)$



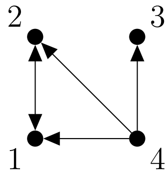
$D_4(5,11)$



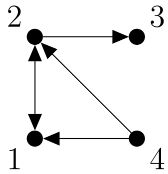
$D_4(5,12)$



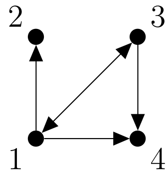
$D_4(5,13)$



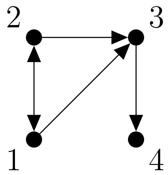
$D_4(5,14)$



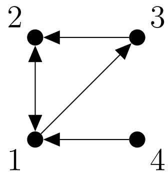
$D_4(5,15)$



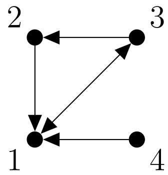
$D_4(5,16)$



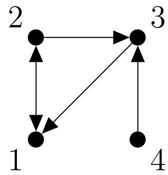
$D_4(5,17)$



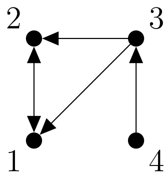
$D_4(5,18)$



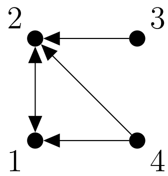
$D_4(5,19)$



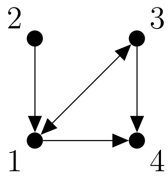
$D_4(5,20)$



$D_4(5,21)$



$D_4(5,22)$



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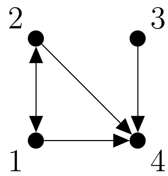
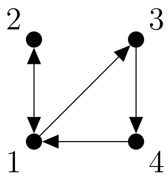


$D_4(5,24)$



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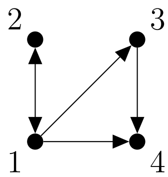
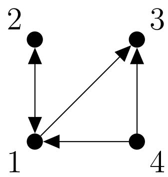
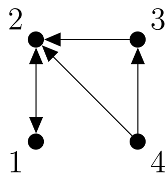


$D_4(5,26)$

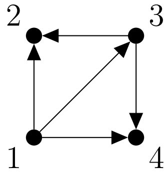


$D_4(5,27)$

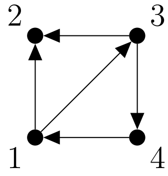


$D_4(5,28)$

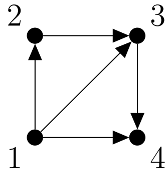
Digraphs of order 4 and 5 arcs: $D_4(5, 29 - 38)$



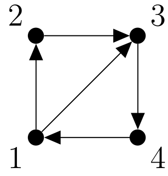
$D_4(5, 29)$



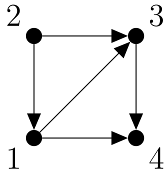
$D_4(5, 30)$



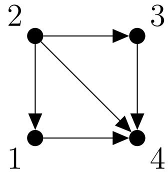
$D_4(5, 31)$



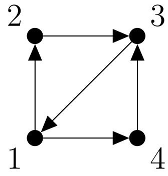
$D_4(5, 32)$



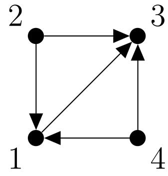
$D_4(5, 33)$



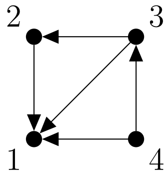
$D_4(5, 34)$



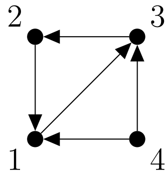
$D_4(5, 35)$



$D_4(5, 36)$

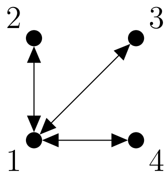


$D_4(5, 37)$

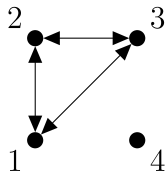


$D_4(5, 38)$

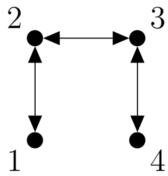
Digraphs of order 4 and 6 arcs: $D_4(6, 1 - 12)$



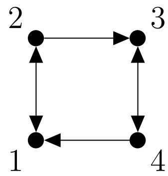
$D_4(6, 1)$



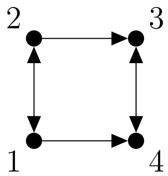
$D_4(6, 2)$



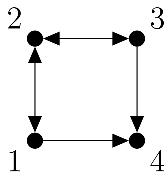
$D_4(6, 3)$



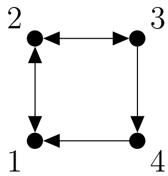
$D_4(6, 4)$



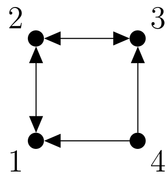
$D_4(6, 5)$



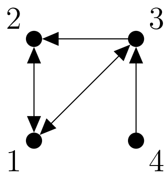
$D_4(6, 6)$



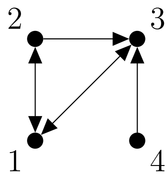
$D_4(6, 7)$



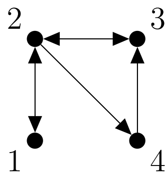
$D_4(6, 8)$



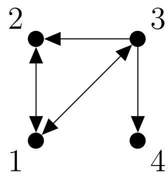
$D_4(6, 9)$



$D_4(6, 10)$

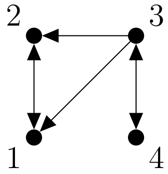


$D_4(6, 11)$

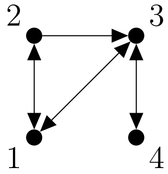


$D_4(6, 12)$

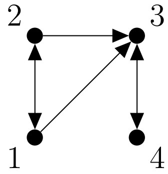
Digraphs of order 4 and 6 arcs: $D_4(6, 13-40)$



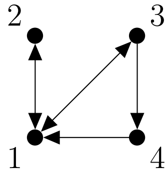
$D_4(6,13)$



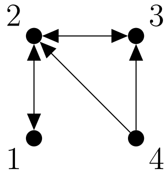
$D_4(6,14)$



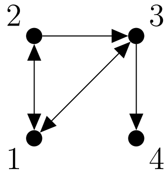
$D_4(6,15)$



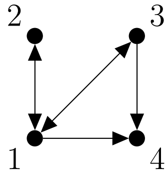
$D_4(6,16)$



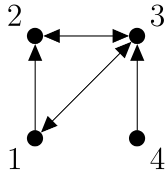
$D_4(6,17)$



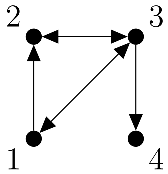
$D_4(6,18)$



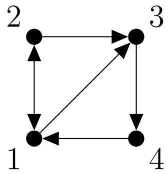
$D_4(6,19)$



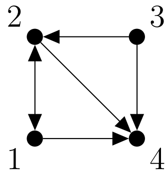
$D_4(6,20)$



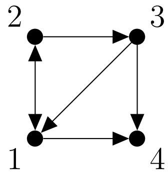
$D_4(6,21)$



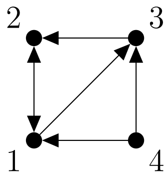
$D_4(6,22)$



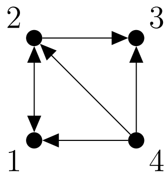
$D_4(6,23)$



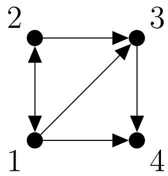
$D_4(6,24)$



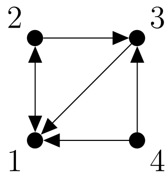
$D_4(6,25)$



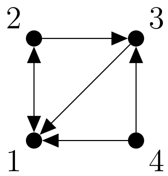
$D_4(6,26)$



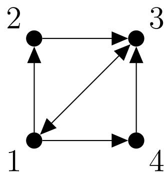
$D_4(6,27)$



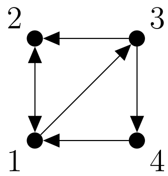
$D_4(6,28)$



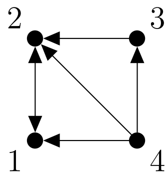
$D_4(6,29)$



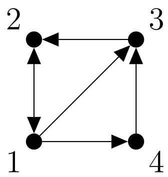
$D_4(6,30)$



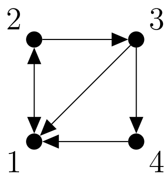
$D_4(6,31)$



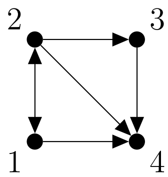
$D_4(6,32)$



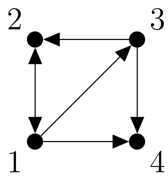
$D_4(6,33)$



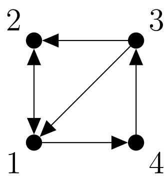
$D_4(6,34)$



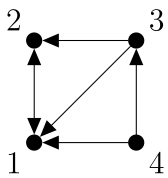
$D_4(6,35)$



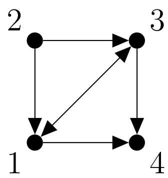
$D_4(6,36)$



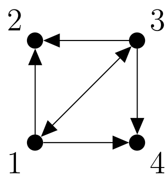
$D_4(6,37)$



$D_4(6,38)$

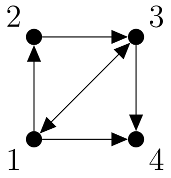


$D_4(6,39)$

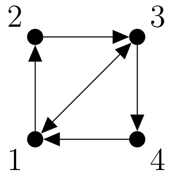


$D_4(6,40)$

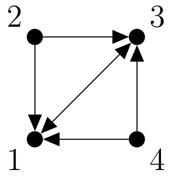
Digraphs of order 4 and 6 arcs: $D_4(6, 41-48)$



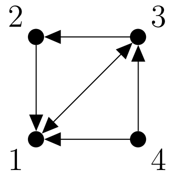
$D_4(6,41)$



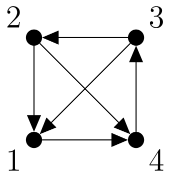
$D_4(6,42)$



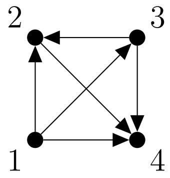
$D_4(6,43)$



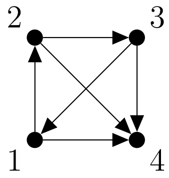
$D_4(6,44)$



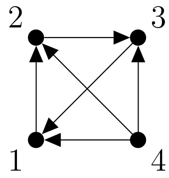
$D_4(6,45)$



$D_4(6,46)$

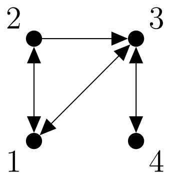


$D_4(6,47)$

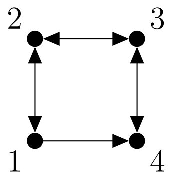


$D_4(6,48)$

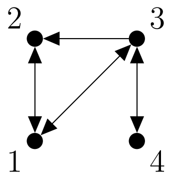
Digraphs of order 4 and 7 arcs: $D_4(7, 1-16)$



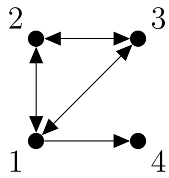
$D_4(7,1)$



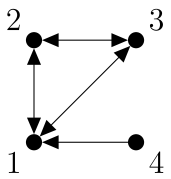
$D_4(7,2)$



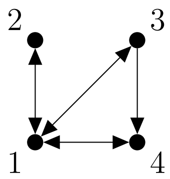
$D_4(7,3)$



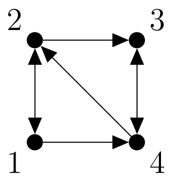
$D_4(7,4)$



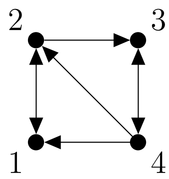
$D_4(7,5)$



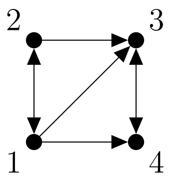
$D_4(7,6)$



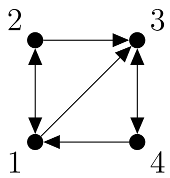
$D_4(7,7)$



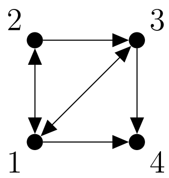
$D_4(7,8)$



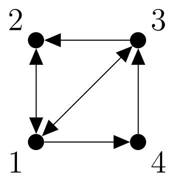
$D_4(7,9)$



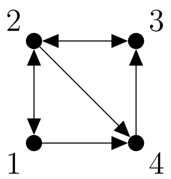
$D_4(7,10)$



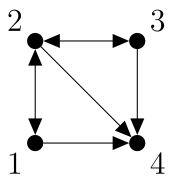
$D_4(7,11)$



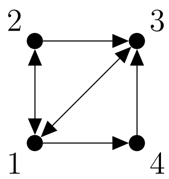
$D_4(7,12)$



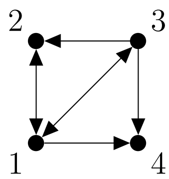
$D_4(7,13)$



$D_4(7,14)$

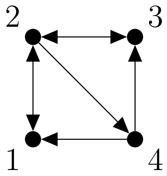


$D_4(7,15)$

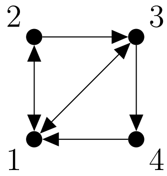


$D_4(7,16)$

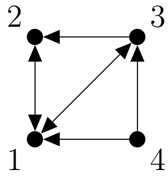
Digraphs of order 4 **and 7 arcs:** $D_4(7, 17 - 38)$



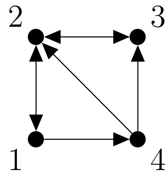
$D_4(7, 17)$



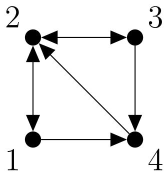
$D_4(7, 18)$



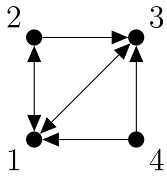
$D_4(7, 19)$



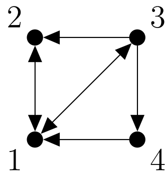
$D_4(7, 20)$



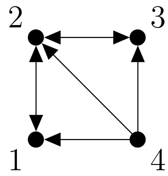
$D_4(7, 21)$



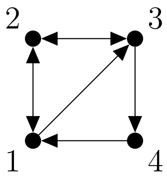
$D_4(7, 22)$



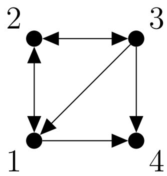
$D_4(7, 23)$



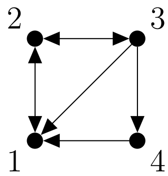
$D_4(7, 24)$



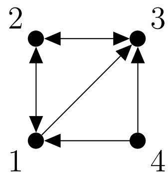
$D_4(7, 25)$



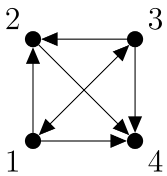
$D_4(7, 26)$



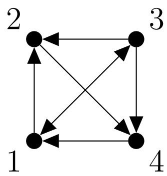
$D_4(7, 27)$



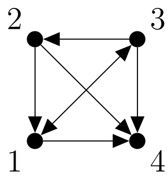
$D_4(7, 28)$



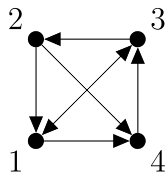
$D_4(7, 29)$



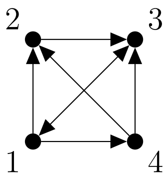
$D_4(7, 30)$



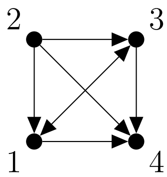
$D_4(7, 31)$



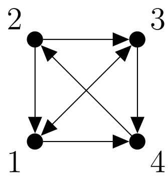
$D_4(7, 32)$



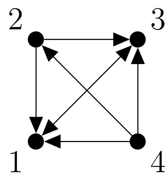
$D_4(7, 33)$



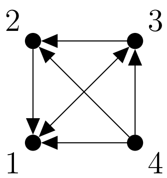
$D_4(7, 34)$



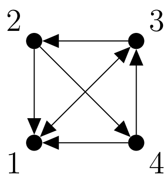
$D_4(7, 35)$



$D_4(7, 36)$

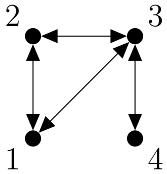


$D_4(7, 37)$

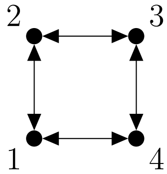


$D_4(7, 38)$

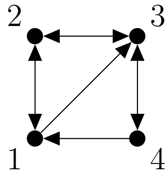
Digraphs of order 4 and 8 arcs



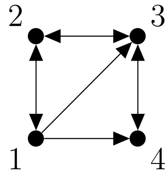
$D_4(8,1)$



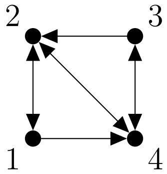
$D_4(8,2)$



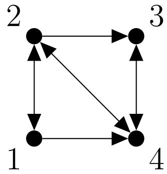
$D_4(8,3)$



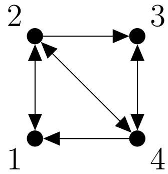
$D_4(8,4)$



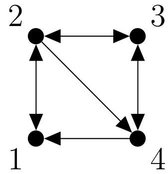
$D_4(8,5)$



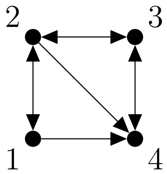
$D_4(8,6)$



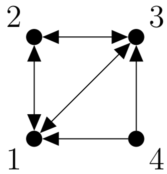
$D_4(8,7)$



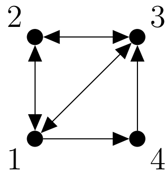
$D_4(8,8)$



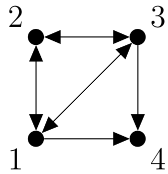
$D_4(8,9)$



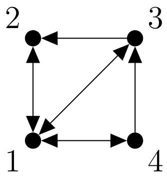
$D_4(8,10)$



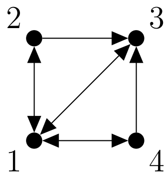
$D_4(8,11)$



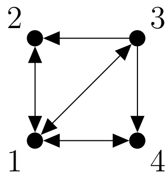
$D_4(8,12)$



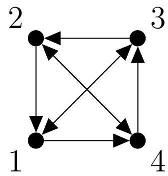
$D_4(8,13)$



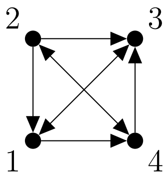
$D_4(8,14)$



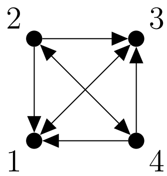
$D_4(8,15)$



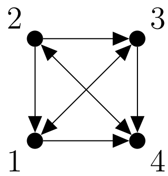
$D_4(8,16)$



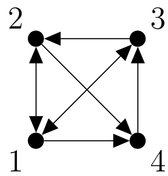
$D_4(8,17)$



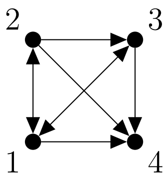
$D_4(8,18)$



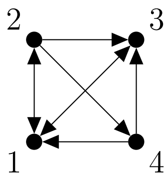
$D_4(8,19)$



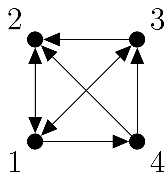
$D_4(8,20)$



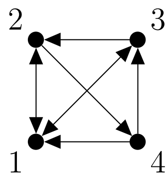
$D_4(8,21)$



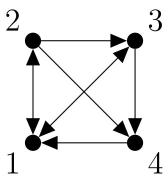
$D_4(8,22)$



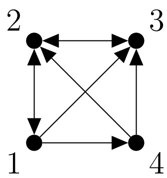
$D_4(8,23)$



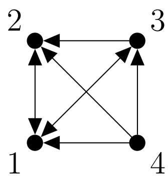
$D_4(8,24)$



$D_4(8,25)$

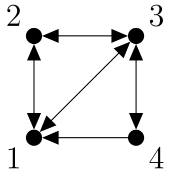


$D_4(8,26)$

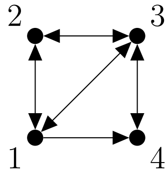


$D_4(8,27)$

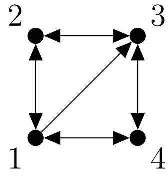
Digraphs of order 4 and at least 9 arcs



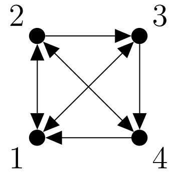
$D_4(9,1)$



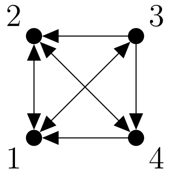
$D_4(9,2)$



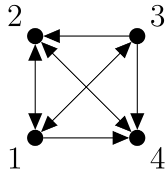
$D_4(9,3)$



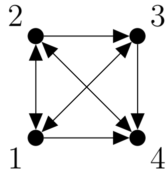
$D_4(9,4)$



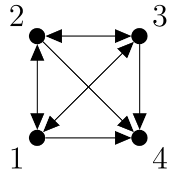
$D_4(9,5)$



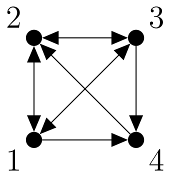
$D_4(9,6)$



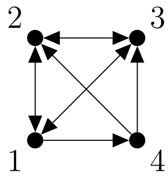
$D_4(9,7)$



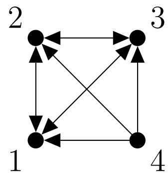
$D_4(9,8)$



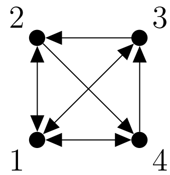
$D_4(9,9)$



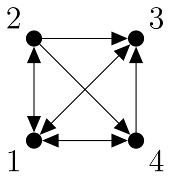
$D_4(9,10)$



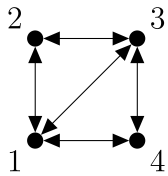
$D_4(9,11)$



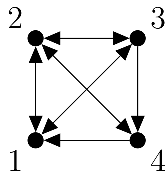
$D_4(9,12)$



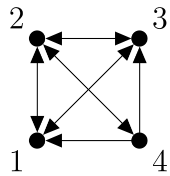
$D_4(9,13)$



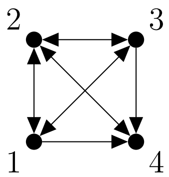
$D_4(10,1)$



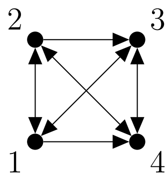
$D_4(10,2)$



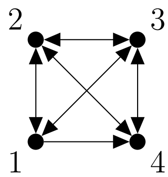
$D_4(10,3)$



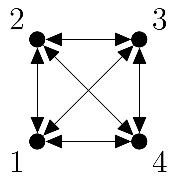
$D_4(10,4)$



$D_4(10,5)$



$D_4(11,1)$



$D_4(12,1)$