RESEARCH PROPOSAL

1. Background

The central goal of the proposed research is a proof of Cherlin's Conjecture for finite binary primitive groups [5].

Conjecture 1. *A finite primitive binary permutation group must be one of the following:*

1. *a symmetric group* $\mathrm{Sym}(n)$ *acting naturally on* n *elements;*
2. *a cyclic group of prime order acting regularly on itself;*
3. *an affine orthogonal group* $V \cdot \mathrm{O}(V)$ *with* V *a vector space over a finite field equipped with an anisotropic quadratic form, acting on itself by translation, with complement the full orthogonal group* $\mathrm{O}(V)$*.*

We need to clarify what it means for a group $G \leq \mathrm{Sym}(\Omega)$ to be called *binary*: first, define a pair $(x_1, \ldots, x_N)$ and $(y_1, \ldots, y_N)$ in Ω^N to be *k-equivalent* for some $k \leq N$ if for every $J = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}$ there exists $g_J \in G$ such that $x_j^{g_J} = y_j$ for every $j \in J$. Now the *relational complexity* of G is defined to be the least k such that whenever two N-tuples are k-equivalent, they are necessarily N-equivalent (for any integer $N \geq 2$); G is called *binary* if its relational complexity is equal to 2.

Note that an alternative (equivalent) definition of a binary permutation group is given below in Definition 3; this definition is harder to state but is (as we shall see) more obviously connected to important questions in the literature.

Cherlin himself gave a proof of the conjecture when G is of "affine type" Wiscons then made use of the O'Nan–Scott–Aschbacher theorem to study various other cases and showed that Conjecture 1 reduces to the following statement concerning almost simple groups [25].

Conjecture 2. *If* G *is a finite binary almost simple primitive group on* Ω*, then* $G = \mathrm{Sym}(\Omega)$*.*

We propose to work as a three-person investigating team: the PI (Gill, South Wales); the Co-I (Liebeck, Imperial); and the Visiting Researcher (Spiga; Milano–Bicocca) to prove Conjecture 2 and, by so doing, to prove Conjecture 1.

1.1. **Model theory context.** Cherlin's conjecture arises from model theory considerations rooted in Lachlan's theory of finite homogeneous relational structures (see, for instance, [17, 19]).

Here a finite relational structure is a generalization of the notion of a directed graph: a directed graph can be thought of as a pair (V, E) where V is a finite set of vertices and E is a set of 2-tuples with entries in V. Now a *relational structure* $\mathcal{R}$ is a tuple $(V, R_1, \ldots, R_\ell)$ where V is a finite set of vertices and each *relation* R_i is a set of k_i tuples for some integer $k_i \geq 2$. For each $i = 1, \ldots, \ell$, the integer k_i is the *type* of the relation R_i; the maximum of the integers k_i is the *type* of the structure $\mathcal{R}$.

The structure $(V, R_1, \ldots, R_\ell)$ is *homogeneous* if, whenever one has $V_1, V_2 \subset V$ such that the induced relational structures on V_1 and V_2 admit an isomorphism $\varphi : V \to V$, then there is an automorphism $\theta : V \to V$ such that $\theta|_{V_1} = \varphi$. In other words, the structure is homogeneous if any local symmetry of the structure extends to a global symmetry. This notion of homogeneity has been studied in many contexts (graphs, geometries, groups,...); Aschbacher, for instance, calls it "The Witt property " due to its connection with some fundamental results of Witt concerning formed spaces [2].

Lachlan's theory implies the existence of a hierarchy of classification theorems for the universe of homogeneous relational structures; this hierarchy is parametrized by the type of the structure; thus, for instance, the first family of theorems in this hierarchy concerns relational structures $(V, R_1, \ldots, R_\ell)$ of type 2 – these are the *binary homogeneous relational structures.* Furthermore each family itself is parametrized by the number of relations in the structure: so the first theorem in the binary family is a result of Lachlan himself [18] that classifies finite binary relational structures with one relation; in other words finite simple homogeneous directed graphs.

It is important to note that Lachlan's theory does not provide the statement of each of these theorems, instead it tells us the "form" that each theorem will take. Thus the form of the ℓ-th theorem in the $(k-1)$-th family in the hierarchy asserts "*A finite homogeneous relational structure of type k with at most ℓ-relations lies in one of a number of infinite families, or else is one of a finite number of sporadic individuals*". And the theory tells us more: namely that each of these sporadic individuals will at some point crop up in an infinite family in one of the later theorems (i.e. in a theorem from family k' for some $k' > k-1$).

Lachlan's theory, then, concerns the behaviour of "sporadicity" in this universe with respect to the type parameter. Of course, one would like to know the precise statement of each of these theorems, starting with the first family, on binary homogeneous relational structures. We shall see in a moment that this family of theorems is closely linked to the statement of Cherlin's conjecture.

We should note, finally, that one could turn the hierarchy on its side and, instead of considering families defined via bounds on type, consider families defined via bounds on the number of relations. This is a rather different problem and it has received considerable attention in the literature, with one of the most celebrated results in this direction due to a team including the Co-I [15].

1.2. **Group theoretic implications.** In the previous section we explained how Lachlan's theory describes a "slicing up" of the universe of finite homogeneous relational structures. In [5], Cherlin changes the perspective of this theory so that it is, instead, seen as a theory concerning the universe of finite transitive permutation groups.

This change of perspective rests on a definition first proposed in [6].

Definition 3. Suppose that G is a transitive permutation group on a set Ω; we say that a relational structure $\mathcal{R} = (\Omega, R_1, \ldots, R_t)$ is *compatible* with this action if the action of $\mathrm{Aut}(\mathcal{R})$ on Ω is permutation isomorphic to the action of G on Ω. Now the *relational complexity* of G is defined to be the minimum type of a homogeneous relational structure that is compatible with the given action; in particular G is a *binary* permutation group if it has relational complexity 2.

It is a simple matter to check that this definition is equivalent to the one given at the start and, moreover, that when Ω is finite, the relational complexity of the action is a positive integer less than $|\Omega|$. With this definition, one sees that Lachlan's hierarchy of classification theorems doesn't just pick up all finite homogeneous relational structures, but in fact picks up all finite transitive permutation groups.

It now becomes clear that the first theorem in Lachlan's hierarchy can be thought of not as a statement about binary relational structures, but about binary permutation groups. What is more these are precisely the objects to which Cherlin's conjecture pertains. However there is one caveat: Cherlin's conjecture only describes those binary permutation groups for which the action on vertices is *primitive*; to obtain the full statement of Lachlan's first theorem, one would need to also consider imprimitive actions.

We are almost ready to discuss Cherlin's conjecture directly, but first we should clarify why the placement of a given permutation group in Lachlan's hierarachy is so very significant. It is an established principle of permutation group theory that if a group G acts *homogeneously* on an object like a relational structure, then a study of the object in question will throw a searching light on the group G and on the action; as Aschbacher says, "... in most categories few objects have the Witt property; those that do are very well behaved indeed." [2]

Examples of this "homogeneity" avenue into the understanding of group actions include Gardiner, Sheehan, Enomoto and Lachlan's classical work on homogeneous graphs [9, 10, 18, 23], as well as variations including set-homogeneous graphs [14] and k-homogeneous graphs [13]; one also has Devillers' work on homogeneous Steiner systems [8] as well as follow-up work by Webb *et. al.* (see EPSRC grant `EP/M016242/1`) and so on. It is also the standard way in to the study of classical groups and polar spaces via Witt's Lemma (see the exposition in Aschbacher [2], or in any standard text on these groups and geometries).

Much of the aforementioned work can be seen as fitting into the scheme outlined by Lachlan. Indeed Lachlan's theory brings an "organisation" to these disparate studies of homogeneity, and provides a deep connection between them.

1.3. **The status of the conjecture.** As described at the start, Cherlin's original conjecture has been reduced to Conjecture 2, which is a statement about finite almost simple groups. Now the Classification of Finite Simple Groups tells us that, to prove Conjecture 2, one must deal with three families of groups: the alternating groups, the sporadic groups, and the groups of Lie type. The first two of these families have been dealt recently with by the PI, Spiga and various co-authors, and we now describe this work.

Alternating groups. Conjecture 2 was proven for the alternating groups by the PI and Spiga in [12]. The following theorem is the statement we need:

Theorem 4. *Let G be a finite almost simple primitive group on Ω with socle an alternating group. If G is binary, then $G = \mathrm{Sym}(\Omega)$.*

The proof of Theorem 4 rested on the notion of a *beautiful subset*; this is a subset Λ of Ω for which the setwise-stabilizer is 2-transitive but does not contain the $\mathrm{Alt}(\Lambda)$. This idea was developed further in [11] where it was also connected to a classical concept in group theory, that of 2*-closure*: the 2-closure of a permutation group $G \leq \mathrm{Sym}(\Omega)$ is the largest subgroup of $\mathrm{Sym}(\Omega)$ that has the same orbits on $\Omega \times \Omega$ as G. If G is equal to its 2-closure, then we say that G is 2-closed, and the study of such permutation groups goes back many years – see, for instance [24].

The connection between 2-closure and Cherlin's conjecture arises from the fact that if Ω has a subset Λ for which the setwise stabilizer G_Λ is not 2-closed, then the action of G on Ω is not binary. (Beautiful subsets can now be seen to be examples of subsets for which the setwise stabilizer G_Λ is not 2-closed.)

Sporadic groups. Conjecture 2 was proved for the almost simple sporadic groups by Dalla Volta, the PI and Spiga in [7]. The methods used here were a little different, as the problem was a largely computational one: there were only finitely many actions to consider and so the problem could be attacked with the computer algebra system `magma` [3].

Groups of Lie type. Conjecture 2 remains open for the groups of Lie type, although some progress has been made. Specifically, the connection between binary actions and 2-closure described above was exploited by the PI, Hunt and Spiga in [11] to prove Conjecture 2 for almost simple groups of Lie type of rank 1, i.e. groups with socle $\mathrm{PSL}_2(q)$, ${}^2\mathrm{B}_2(q)$, ${}^2\mathrm{G}_2(q)$ or $\mathrm{PSU}_3(q)$.

We are left with the job of proving Cherlin's conjecture for finite almost simple groups of Lie type of rank at least 2. Many of the methods developed in [7, 11, 12] can be applied to this task; indeed the connection with 2-closure has already been exploited in [12] (via "beautiful subsets") to resolve the conjecture for the $\mathcal{C}_1$-actions of the classical groups. Thus the efficacy of these methods in this context has been demonstrated although the task that remains is considerable, and many more ideas will be needed (see §4 below).

2. Academic Impact

A proof of Conjecture 1 would complete the first major problem that was highlighted in this area by Cherlin in his 2000 paper [5]. It would also, as described above, represent a significant step towards proving the first family of theorems proposed by Lachlan in 1986 [19].

Perhaps more significantly the recent progress made by the PI and Spiga into Conjecture 1 have thrown up a number of techniques that have immediate and obvious application.

Imprimitive actions. The methods developed in [7, 11, 12] rely on primitivity in the weak sense that the assumption of primitivity allows one to restrict one's attention to a smaller class of actions: the primitivity assumption does not enter the analysis of any given action.

To study imprimitive actions, then, one has two options: either one tries to prove some kind of "reduction theorem" that allows one to make use of the primitive case. In another direction, one tries to "organise" imprimitive actions in some sensible way, so that the techniques developed in the primitive analysis can be applied directly.

To illustrate the latter direction, we refer to a result in [7] pertaining to $r_\ell(G)$, the number of orbits of G on $\Omega^{(\ell)}$ (here $\Omega^{(\ell)}$ is the set of all ℓ-tuples of distinct elements of Ω):

Lemma 5. *If G is transitive and binary, then $r_\ell(G) \leq r_2(G)^{\ell(\ell-1)/2}$ for each $2 \leq \ell \in \mathbb{Z}$.*

We believe that this sort of result can be applied effectively to finite imprimitive actions, and in so doing we have a reasonable prospect of making further progress towards Lachlan's first theorem.

Small relational complexity. Again, the new methods have immediate and easy application to the problem of classifying transitive permutation groups of small relational complexity, for instance weakening the "binary" assumption in Cherlin's conjecture to "ternary". By way of illustration, Lemma 5 has an analogue in this setting, the statement being " If G is transitive and ternary, then $r_\ell(G) \le r_3(G)^{\ell(\ell-1)/2}$ for each $3 \le \ell \in \mathbb{Z}$." Another example pertains to the connection between "binariness" and 2-closure discussed above; one has an analogous connection between "ternariness" and 3-closure, and so on.

These insights open up the possibility of being able to populate the lower tiers of Lachlan's hierarchy in a reasonably complete manner. The insight this could yield into the study of permutation groups is significant: it would efficiently associate structures with each permutation group such that the group acts homogeneously; it would allow us to understand this process by which sporadic examples become absorbed into infinite families (one naturally wonders, for instance, about the rate at which this happens).

Calculating relational complexity for important families. The calculation of the precise value of the relational complexity of any given group action can be surprising delicate and difficult. Even obtaining tight bounds has proved a difficult task. To appreciate the technical hurdles that such calculations throw up, we refer to [6] where an exact formula is given for the action of $\mathrm{Sym}(n)$ on k-subsets of $\{1, \dots, n\}$, as well as various bounds for the actions of certain wreath products.

There are many well-studied families of permutation groups for which there do not exist any non-trivial bounds on the relational complexity. One naturally wonders, for instance, about the actions of the finite classical groups on their associated projective/ polar spaces: [12, Theorem B] gives a lower bound of 3 for these actions (unless $G \cong \mathrm{PGL}_2(4)$), and there is some discussion of how to improve this lower bound to 4. This seems unsatisfactory though: surely one should be able to establish a lower bound as a function of the rank and field. And, likewise, for an upper bound.

2.1. **Collaboration.** The proposed team of three researchers – Gill, Liebeck and Spiga – has all of the necessary expertise to write a proof of Conjecture 1.

Specifically, Gill and Spiga have led the recent progress in resolving Conjecture 1 by exploiting the Classification of Finite Simple Groups. They have developed the new methods that allow one to understand relational complexity using classical notions from permutation group theory like 2-closure, bounds on $r_\ell(G)$, the structure of subset stabilizers and so on. More such connections will need to be made to close out the proof, and Gill and Spiga are in the optimal position to make these advances.

The progress made by Gill and Spiga means that Conjecture 1 has been reduced to a question about the primitive actions of the finite groups of Lie type. Liebeck is one of the preeminent authorities in this area: the book [16] is the definitive text on these actions for the case where the group is classical; if the group is exceptional, then Liebeck's work with Seitz is the basis for our current understanding (see, for instance [22], and the references therein).

In short, the expertise brought by the team is tailor-made to achieve our stated goal.

3. Research Hypothesis and Objectives

The timeliness of the proposed research is illustrated by the recent progress made by the PI and Spiga: the question of classifying binary permutation groups has been open since relational complexity was first defined in 1996. Progress towards this goal was frustratingly slow until, first, Cherlin dealt with the affine case, second, Wiscons provided the important reduction to almost simple groups and, third, the PI and Co-I were able to use the Classification of Finite Simple Groups to effectively address what remains. We have arrived at the point where completing a full proof of Conjecture 1 is a reasonable prospect.

The novelty of the proposed approach is in its combination of a known successful strategy (the methods developed in [7, 11, 12]) combined with a deep understanding of the theory of the finite simple groups, focusing specifically on the almost simple groups of lie type.

We feel very confident that we can deliver our primary goal, a proof of Cherlin's conjecture; we are confident, too, that the extensions described in §2 will transform the study of relational complexity from a frustrating collection of intractable calculations into the field that it promises to be: a rich source of insight into the structure of finite permutation groups.

4. Programme and Methodology

The proposed research is of a pure mathematical nature, so will proceed through the careful application of new methods for binary groups together with the technical theory of finite groups of Lie type.

Studying the primitive actions of these groups amounts to studying their maximal subgroups. This allows the programme to split naturally into a number of components.

4.1. **Classical groups: geometric actions.**

The maximal subgroups of the almost simple classical groups were classified by Aschbacher [1] into nine families; this classification was then refined by Kleidman and the Co-I [16].

The first eight families in this classification are all defined with respect to the action of the group on its associated natural module. These are the "geometric actions", they are all explicitly described and their complete structure is understood. One of these families has already been dealt with in [12] – the $\mathcal{C}_1$ family – and it is expected that the methods used there ("beautiful subsets") will carry over directly in many settings. The following statement, for instance, can be deduced using the methods of [12]: *Let G be an almost simple primitive permutation group on the set Ω with socle isomorphic to $\mathrm{PSL}_n(q)$ with $n \geq 3$. Then all of the $\mathcal{C}_2$ actions admit beautiful subsets and so are not binary.*

Many of the other cases are much less straightforward: for instance the $\mathcal{C}_3$ family in $\mathrm{PSL}_n(q)$ includes the normalizers of Singer cycles. The corresponding actions do not admit beautiful subsets in general, and so new and different methods will be needed.

4.2. **Classical groups: $\mathcal{C}_9$ actions.**

The final family in Aschbacher's classification consists of almost simple groups embedded irreducibly in our classical group. Here the classification is less explicit, and one must bring to bear techniques from representation theory to limit the number of possible actions.

Instead of dealing with these actions one at a time, one must try and leverage the irreducibility of the embedding to yield general information. This has been done in many contexts so that we now, for instance, have a lot of information about the size of the $\mathcal{C}_9$-subgroups [20], the number of conjugacy classes of the $\mathcal{C}_9$-subgroups [21], fixed-point ratios [4] and so on. The aim is to combine this kind of information with appropriate sufficient conditions for an action to be non-binary in order to obtain the result; this is reminiscent of the fixed-point arguments that were combined with [12, Lemma 3.7] to deal with certain primitive actions of the alternating groups.

4.3. **Exceptionals.**

The primitive actions of the almost simple exceptional groups are classified similarly to the classicals in that there is a natural split between geometric subgroups and almost simple subgroups. The advantage one has here, though, is that rank is bounded in the exceptionals and so one can (in theory) write down explicitly all of the irreducible almost simple subgroups for each family of exceptional groups. This has not *quite* been achieved – there are some possible subgroups for which embeddings are either not known or not fully understood.

The first task with the exceptional groups is to deal with the parabolic subgroups which we expect to yield to the same approach as was taken to the classical parabolic subgroups in [12]. After this, the task is much less straightforward as it can be very difficult to "get one's hands on" subgroups of the exceptional groups; we will need to combine a good technical understanding of these subgroups, along with significant new ideas concerning binary actions.

5. National Importance

The proposed research fits within the *Mathematical Sciences* section of the EPSRC portfolio and, within this, in the *Algebra* theme. Moreover, as we have seen, there are strong connections to the theme of *Logic and combinatorics*. Indeed, given its intellectual ancestry in Lachlan's model theory and its strong and concrete connection to permutation group theory, the proposed research stands to be of significant interest to a wide community of mathematicians.

The study of permutation group theory has a long history with connections to every area of mathematics. In particular, a large number of mathematicians have focused on questions pertaining to the "homogeneity" of particular actions; we mention, for example Evans (Imperial), Gray (East Anglia), Macpherson (Leeds), Praeger (Western Australia), Tent (Münster).

The model-theoretic study of "classification of structures" (of which this work forms a part) has attracted the attention of many famous individuals, including Cherlin himself (Rutgers), Hrushovski (Oxford and Jerusalem), and Shelah (Rutgers and Jerusalem; note that Lachlan's work on relational structures can be thought of as an application of various insights of Shelah). Progress with Cherlin's conjecture will be of interest to all of these individuals, and others across the model theory community.

We should emphasise that the connections between permutation group theory and model theory have proved fruitful in the past and has been of particular interest in UK mathematics, thanks in no small part to the work of the Co-I (see, for instance, his collaborations with Macpherson (Leeds)). Problems to do with homogeneous structures have been a regular focus in meetings across the UK: for instance the 2011 four day meeting in Leeds and, very recently, the Dame Kathleen Ollerenshaw Combinatorics Day at the University of Manchester in July 2017. This last meeting was devoted entirely to Cherlin's conjecture, including a focus on the recent progress of the PI and Spiga.

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TRACK RECORD

The proposed research area is in the area of finite group theory. The main focus of the research is to prove what remains of Cherlin's conjecture for primitive binary permutation groups. This conjecture proposes a classification of those primitive permutation groups that have relational complexity equal to 2. Due to work of Cherlin himself and Wiscons, the conjecture reduces to a question concerning almost simple groups.

The proposed three-person research team – the PI (Gill), the Co-I (Liebeck), and the Visiting Researcher (Spiga) – is ideally equipped: all three are experienced researchers in finite group theory with specific expertise giving the best possible chance of achieving the stated goal.

The PI and Spiga's results on Cherlin's Conjecture. An *almost simple group* is a finite group that has a unique minimal normal subgroup N and such that N is a non-abelian simple group. The Classification of Finite Simple Groups asserts that N must be either alternating, sporadic or a finite group of Lie type. Thus, to prove Cherlin's conjecture one must deal with each of these families in turn.

The PI and Spiga (along with various co-authors) have already made considerable progress: in [4] they deal with the case where N is alternating; in [2] they deal with the case where N is alternating; finally in [3, 4] two significant subcases of the groups of Lie type are dealt with.

The three just-cited papers represent the most significant current contribution to our understanding of the relational complexity of actions of finite simple groups. Because the structure of the finite simple groups varies dramatically from family to family, these papers each required substantial new perspectives on the notion of relational complexity for finite permutation groups; these new perspectives were then applied to the families in question, tailored to deal with the technical obstacles inherent in dealing with these difficult groups.

These results of the PI and Spiga already represent success stories in this area. They give solid evidence that the PI and Spiga are well-placed to deal the remaining family – the finite groups of Lie type – and to thereby complete a full proof of the conjecture.

The Co-I's work on the finite groups of Lie type. The primary obstacle to completing the proof of Cherlin's conjecture is likely to be the technical difficulty of handling the finite groups of Lie type. The Co-I has a track-record of world-class research in this area.

Understanding the primitive actions of the finite groups of Lie type is equivalent to understanding the structure of their maximal subgroups. This task splits naturally into two parts, according to whether the group of Lie type is *classical* or *exceptional*.

In the classical case, the original structure statement is due to Aschbacher [1]. The definitive work on this structure statement is due to the Co-I and Kleidman [6]; in this work they sharpen and extend Aschbacher's result, giving forensic detail concerning the structure and behaviour of the subgroups in question. This work will be the foundation on which the proposed treatment of the classical groups is built (and, indeed, the partial case already dealt with by the PI and Spiga in [4] uses the framework given in [6]).

In the exceptional case, the original structure statement (an analogue of the aforementioned result of Aschbacher) is due to the Co-I and Seitz [7]. A great deal of work has since gone in to sharpening this result until it reaches the level of detail found in [6] for the classicals; this work has been led by the Co-I and Seitz, and is spread over a variety of articles; a summary can be found in, for instance, [8].

The Co-I's work in model theory. As alluded to in the description of the proposed research, Cherlin's conjecture can be thought of as a natural complement to existing work in the literature that seeks to understand homogeneous actions on relational structures where the number of types in the structure is limited. In this direction one of the most significant results in the literature is due to the Co-I and co-authors [5].

As well as being complementary in perspective, this result is note-worthy here as it involves similar techniques to those proposed in this project: the Classification of Finite Simple Groups is used to "deconstruct" an otherwise intractible problem into a series of smaller problems that can be attacked using a technical understanding of specific families of permutation groups. The success of the Co-I in this endeavour is additional solid evidence of success in this area.

General track record. The proposed three-person team has a strong general track record of research in group theory, over and above the directly relevant research mentioned above.

The PI (Gill) has contributed to the study of permutation group actions on finite geometries (especially finite projective planes and designs); he has worked on "growth in groups" following on from the breakthrough work of Helfgott; he has worked on "width" questions in groups together with Pyber and others, and inspired by a conjecture due to the Co-I, Nikolov and Shalev; more recently, he has helped to develop a new theory of "Conway groupoids" inspired by Conway's famous M_{13}-construction. All of this research lies in the realm of finite group theory, with much of it making use of the Classification of Finite Simple Groups.

The Co-I (Liebeck) has been one of the leaders in the study of the subgroup structure of the finite simple groups over the past three decades. He is one of the pioneers of modern probabilistic group theory, particularly in its application to simple groups, and has achieved some spectacular recent applications of representation theory, such as the proof of the 60 year old Ore conjecture.

The Visiting Researcher (Spiga) is one of the most prominent researchers in the study of group actions on graphs and on other combinatorial objects. His main expertise is within finite primitive groups and applications of the O'Nan-Scott-Aschbacher theorem for investigating symmetries of combinatorial structures. Combining the recent developments in subgroup growth with p-local analysis in simple groups, he has proved some important special cases of the 50 year old Weiss conjecture on locally primitive graphs.

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The University of South Wales, which supports the PI, does not have the same strong track record in algebra, although that has started to change since the PI obtained an EPSRC First Grant: the PI and Hunt (also of South Wales) have collaborated with Spiga in [3] (this work is discussed in the *Proposed Research* document); in addition, South Wales have funded a PhD in support of the successful First Grant bid, and have committed to do the same in the event of the current proposal being successful.

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