Section 3.

(E3.1) Verify that the function described in Example 10 is an action, i.e. that (A1) and (A2) hold.

(E3.2) Verify that the function described in Exercise 11 is an action.

(E3.3) Suppose that we changed the function described in Exercise 11 from $v{\cdot}g$ to $g{\cdot}v$ (so, for instance, if V is finite-dimensional we consider v as a column vector and use matrix multiplication). Show that this is not an action. Can you find a 'natural' adjustment to this definition so that it becomes an action?

(E3.4) Give sufficient conditions such that

$$\langle A\rangle = \{a_1a_2\cdots a_k \mid k\in\mathbb{Z}^+, a_i\in A\}.$$

Give an example of a set A in a group G for which this inequality does not hold.

(E3.5) Let g and h and $G=\langle g,h\rangle$ be as given in Example 12. Prove that the order of g (resp. h) is n (resp. 2), and that $h^{-1}gh=g^{-1}$. Prove that G is of order $2n$ and that G contains a normal cyclic subgroup C of order n. Prove that every element in $G\backslash C$ has order 2.

(E3.6) Check that the action described in Example 12 is a well-defined action of G on X as an object from **SimpleGraph**.

(E3.7) See if you can define analogues of the categories $\mathbf{G-Set}$ and $\mathbf{G-Set2}$ for which Lemma 3.2 amounts to a statement about equivalence of categories.

(E3.8) Show that G_ω is a subgroup of G for all $\omega\in\Omega$. Show that $G_{(\Omega)}$ is a normal subgroup of G, equal to the kernel of the associated homomorphism ϕ^*.

(E3.9) Suppose that a group G acts on a set Ω. Show that the set of orbits

$$\{\omega^G \mid \omega\in\Omega\}$$

partitions Ω.

(E3.10) let $G=\mathrm{Sym}(\Omega)$ in Example (E1.1). Prove that the action is faithful. Under what conditions is it transitive (resp. semiregular)? Describe the stabilizer of an element of Ω.

(It may be easier to restrict to the case where Ω is finite. In which case we can choose a labelling so that $\Omega=\{1,\dots,n\}$, for a positive integer n.)

(E3.11) let $G=\mathrm{GL}(V)$ in Example (E1.1). Prove that the action is faithful. Under what conditions is it transitive (resp. semiregular)? Describe the stabilizer of the zero vector. Let V be finite-dimensional, choose a basis $\{e_1,\dots,e_n\}$ and describe the stabilizer of e_1.

(E3.12) Consider the action described in Example 12. Prove that the action is both faithful and transitive (and hence the action induces an embedding of D_{2n} in $\mathrm{Aut}(X)$). What are the vertex-stabilizers in this action? When does $D_{2n}=\mathrm{Aut}(X)$?

(E3.13) Verify that (2) holds, thereby completing the proof of Lemma 3.3.

(E3.14) What conditions on H are equivalent to the action of G on $H\backslash G$ being faithful?

(E3.15) Let G be a finite group acting transitively on a set Ω. Show that the average number of fixed points of the elements of G is 1, i.e.

$$\frac{1}{|G|}\sum_{g\in G}|\{\omega\in\Omega \mid \omega^g=\omega\}| = 1.$$

(E3.16) Prove that the map ϕ is a well-defined group homormophism from G to $\mathrm{Aut}(G)$ (and, hence, the action of G on itself by conjugation is an action on itself as an object from **Group**.)

(E3.17) Prove that if g and h are conjugate elements of G, then they have the same order.

(E3.18) Prove that a normal subgroup of G is a union of conjugacy classes of G.

(E3.19) Let N be a normal subgroup of G. Prove that G acts (by conjugation) on N as an object grom **Group**. (In particular, whenever N is a normal subgroup of G, the conjugation action induces a morphism $G\to\mathrm{Aut}(N)$.

(E3.20) Prove Lemma 3.5.

(E3.21) Consider the action of G by conjugation on the set of all subgroups of G. If H is a subgroup of G and $\{H\}$ is an orbit under this action, then what type of subgroup is H?

(E3.22) Prove that if G acts transitively on Ω and G_ω is a stabilizer, then the set of all stabilizers equals the set of all conjugates of G_ω. Under what conditions is the action of G by conjugation on this set of conjugates permutation isomorphic to the action of G on Ω?

(E3.23) Prove that if G is a regular permutation group on Ω then $C_{\mathrm{Sym}(\Omega)}(G)$ is regular.

(E3.24) Prove that if G is a regular permutation group on Ω, then G is permutation isomorphic to $C_{\mathrm{Sym}(\Omega)}(G)$.