5. Primitivity and related notions

(E5.1) For which values of n is the action of D_{2n} on an n-gon, 2-transitive?

(E5.2) Show that, for $k \geq 2$, if an action is k-transitive, then it is $k-1$-transitive.

(E5.3) Let $G = S_n$, the symmetric group on n letters. What is the largest value of k for which G is k-transitive? What about $G = A_n$, the alternating group on n letters?

(E5.4) Prove that if an action is 2-transitive, then it is primitive.

(E5.5) Prove that if an action is transitive and $\sim$ is a G-congruence, then all of the blocks associated with $\sim$ have the same cardinality.

(E5.6) Suppose that G acts imprimitively on Ω. Let Δ be a block associated with the action of G on Ω and let $\omega \in \Delta$. Prove that Δ is a union of orbits of the stabilizer G_ω.

(E5.7) Complete the proof by showing that if G acts transitively on Ω, and if M is a subgroup of G such that $G_\omega < M < G$ for some $\omega \in \Omega$, then the action is imprimitive.

(E5.8) Check that this function is the same as (4) (thereby proving that it defines an action of H on B as an object in Group).

(E5.9) Prove that the group G in Example 16 is equal to

$$\{g \in \mathrm{Sym}(6) \mid i \sim j \Longrightarrow i^g \sim j^g\},$$

where $\sim$ is the G-congruence defined at (6).

(E5.10) Find a copy of $\mathrm{Sym}(2) \wr \mathrm{Sym}(3)$ inside $\mathrm{Sym}(6)$. Describe its action on $[1, 6]$.

(E5.11) Prove the converse to this lemma: If $A \wr G$ acts faithfully on $\Lambda \times \Omega$, then G acts faithfully on Ω and A acts faithfully on Λ.

(E5.12) Our category is called **ImprimTrans**

Objects: Our objects are pairs $(\Omega, \sim)$ where Ω is a finite set and $\sim$ is an equivalence relation for which all equivalence classes have the same size.

Arrows: An arrow $(\Omega, \sim) \to (\Omega', \sim')$ is a function $f : \Omega \to \Omega'$ such that $x \sim y \Longrightarrow f(x) \sim' f(y)$.

1. Prove that **ImprimTrans** is a category.
2. Prove that if X is an object in **ImprimTrans**, then $\mathrm{Aut}(X) \cong \mathrm{Sym}(\Lambda) \wr \mathrm{Sym}(\Delta)$ for some finite sets Λ and Δ.
3. Prove that if G acts on $X = (\Omega, \sim)$ as an object from **ImprimTrans**, then $\sim$ is a G-congruence, and conversely.
4. Consider an alternative category – called **Imprim** in which we drop the condition that equivalence classes all have the same size. What would $\mathrm{Aut}(X)$ look like in this case?

(E5.13) Let H be a primitive subgroup of $\mathrm{Sym}(n)$. Prove that

1. if H contains a transposition, then $H = \mathrm{Sym}(n)$.
2. if H contains a 3-cycle, then H contains $\mathrm{Alt}(n)$.[4]

(E5.14) Let Ω be a finite set of order n, and let $X = (\Omega, \sim)$ (resp. $Y = (\Omega, \sim')$) be an object from **ImprimTrans**. Assume that neither $\sim$ nor $\sim'$ are trivial. Let $H = \mathrm{Aut}(X)$ (resp. $K = \mathrm{Aut}(Y)$) be subgroups of $\mathrm{Sym}(n)$.

1. Use the result of the previous exercise to prove that H is maximal.
2. Are H and K conjugate? How many conjugacy classes of subgroups isomorphic to H does $\mathrm{Sym}(n)$ contain?
3. Describe the intersection of H and $\mathrm{Alt}(n)$.

[4]This is a famous result of Jordan. Its proof is a little tricky.