6. The product action

(E6.1) Consider the product action of the group $\mathrm{Sym}(2) \wr \mathrm{Sym}(3)$ (on a set of size 8). Is this action primitive?

(E6.2) Prove the converse.

(E6.3) Prove the converse.

(E6.4) Let p be a prime, $\ell > 1$ any positive integer. Let

$$C_p = \langle (1, 2, 3, \dots, p) \rangle$$

be a cyclic subgroup of order p in $\mathrm{Sym}(p)$, and consider the wreath product $G = C_p \wr \mathrm{Sym}(\ell)$ in the product action on a set of size p^ℓ. Prove that the action is transitive and imprimitive; calculate the order of the blocks of imprimitivity preserved by G; describe the setwise stabilizer of a block of imprimitivity.

(E6.5) Our category is called **ProductStruct**

Objects: An object is a pair (Ω, θ) where Ω is a finite set and $\theta : \Omega \to \Gamma^\Delta$ is a product structure. Equivalently an object is a direct product $\underbrace{\Gamma \times \cdots \times \Gamma}_{\ell}$ where Γ is a finite set of size k.

Arrows: An arrow is a pair (g, h) where $g : \Omega \to \Omega$ and $h : \Delta \to \Delta$ are functions, and we require that (7) holds.

(1) Prove that **ProductStruct** is a category.
(2) Prove that if X is an object in **ProductStruct**, then $\mathrm{Aut}(X) \cong \mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$.
(3) Prove that if G acts on $X = \Gamma^\ell$ as an object from **ProductStruct**, then $\sim$ is a G-product structure, and conversely.

(E6.6) Let Ω be a finite set of order n and let $X = (\Omega, \theta)$ (resp. $Y = (\Omega, \theta')$) be an object from **ProductStruct**. Let $H = \mathrm{Aut}(X)$ (resp. $K = \mathrm{Aut}(Y)$) be subgroups of $\mathrm{Sym}(n)$. When is H maximal? Are H and K conjugate? How many conjugacy classes of subgroups isomorphic to H does $\mathrm{Sym}(n)$ contain? Describe the intersection of H and $\mathrm{Alt}(n)$.