Classical Groups: Discussion Class

15th October 2013

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Let $G = \mathrm{GL}_n(k)$.

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(D1) Given a basis $\{e_1, \ldots, e_n\}$, the chain of subspaces

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Remark: The group B and any conjugate of B is called a *Borel subgroup* of G.

(D2) Given a basis $\{e_1, \ldots, e_n\}$, the corresponding *frame* is the set

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Let N be the stabilizer in G of the given frame. What is N?

Answer. N is the set of all monomial matrices, that is all matrices with precisely one nonzero entry in each row and column.

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Answer. 1. Let $g \in \mathrm{GL}_n(k)$ and let j be the last row such that $a_{j1} \neq 0$. For each $i < j$ premultiplying by a suitable transvection matrix $x_{ij}(\alpha) \in B$ is the elementary row operation $r_j \mapsto r_i + \alpha r_j$, so we can make a_{j1} the only nonzero entry in the first column.

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Answer.
2. Since A is invertible, there exists $j' \neq j$ such that $a_{j'2} \neq 0$. Take j' to be the last such row. By premultiplying by transvections from B we can make all entries in column 2 except for those in rows j and j', equal to 0.

(D3) Show that $G = \langle B, N \rangle$.

Answer.
3. Repeating this process we obtain a matrix h such that for each column k there is a unique row whose first nonzero entry is in column k. Notice that $h = bg$ where $b \in B$.

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Answer.
4. Now there is a permutation matrix $n \in N$ such that nh is an upper triangular matrix, i.e. $nh = b' \in B$. We conclude that $nbg = b'$, i.e. $g = b'n^{-1}b^{-1}$. Since g was arbitrary we are done.

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Remark: In fact our proof shows that $G = BNB$.

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Remark: Note that H is the group of all diagonal matrices.

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Answer. Let P be the subgroup of permutation matrices, i.e. the set of all matrices with one non-zero entry in each row and column, and all non-zero entries equal to 1.

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Answer. Let P be the subgroup of permutation matrices, i.e. the set of all matrices with one non-zero entry in each row and column, and all non-zero entries equal to 1. Each $n \in N$ can be written as the product of a diagonal matrix in H with a permutation matrix from P. Thus $N = HP$ and, since $P \cap H = 1$, we obtain that

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Now we claim that $P \cong S_n$, the symmetric group on n letters. The isomorphism is given by the map that takes each permutation $\sigma \in S_n$ to the matrix with a 1 in the (i, j)-entry if $i^\sigma = j$ and 0 in all other entries.

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We call $|R|$ the *rank* of the BN-pair.

(D6) Let $R := \{(1,2),(2,3),(3,4),\ldots,(n-1,n)\}$ a generating set of size $n-1$ for the group S_n.
Prove that, with this generating set, (3a) and (3b) are satisfied for $GL_n(k)$, i.e. $GL_n(k)$ has a BN-pair.

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Prove that, with this generating set, (3a) and (3b) are satisfied for $GL_n(k)$, i.e. $GL_n(k)$ has a BN-pair.

Answer. (3b) is an easy matrix calculation. (3a) is a slightly more tricky matrix calculation that I leave for your edification.

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2. Tits has shown that given any group with a BN-pair, we can define a *building* on which G has a natural action. What is more, in this action, G is 'transitive on the pairs consisting of an apartment and a chamber contained in it' [Tit74, 3.2.6].

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3. Conversely Tits has shown that if a group G acts on a building so that it is 'transitive on the pairs consisting of an apartment and a chamber contained in it', then G has a BN-pair [Tit74, 3.11]. Thus the notion of a BN-pair and a building with this level of transitivity are closely linked.

1. Finally Tits has shown that a finite building of ‘irreducible type’ and rank at least 3 is isomorphic to ‘the building of a finite group of Lie type’. What is more such buildings admit transitive actions of the associated groups and we thereby have a full classification of those finite groups with a BN-pair of rank at least 3.

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2. Since the simple classical groups are groups of Lie type, they all have BN-pairs. Can you identify the groups B and N?

Jacques Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, Vol. 386, Springer-Verlag, Berlin, 1974.