EXERCISE SHEET 3

(E60) Prove that the left and right radicals are subspaces.

Answer. We just consider the left radical, as the right is the same. Let v, w be vectors in the left radical and $c, d \in k$. Then, for $y \in V$,

$$\beta(cv + dw, y) = c^\sigma \beta(c, y) + d^\sigma \beta(w, y) = c^\sigma \cdot 0 + d^\sigma \cdot 0 = 0.$$

The result follows immediately.

(E61) Prove that if $\dim V < \infty$, then the left and right radicals have the same dimension. Give a counter-example to this assertion when $\dim V = \infty$.

Answer. If $\dim(V) < \infty$, then we fix a basis for V and let A be the matrix for the form β with respect to this basis. Now the left radical is equal to the null space of A while the right radical is equal to the null space of A^T. Basic linear algebra implies that these have the same dimension.

Let V be the vector space of infinite sequences $(x_1, x_2, x_3, \dots)$ which have only finitely many non-zero entries. Now one can define

$$\beta : V \times V \to k, x \mapsto x^T \cdot A \cdot y,$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & & \ddots & \end{pmatrix}$$

is an 'infinite matrix'. (I haven't defined such an object, but it should be clear what I mean.) It is clear that the left radical is non-trivial - it is equal to

$$\{(x, 0, 0, \cdots) \mid x \in k\}.$$

On the other hand the right radical is certainly trivial. We are done.

(E62) Check that the following map is a duality.

e: perp (1) $$\mathrm{PG}(V) \to \mathrm{PG}(V), U \mapsto U^\perp := \{x \in V \mid \beta(x, y) = 0 \text{ for all } y \in U\}.$$

Answer. That incidence is preserved is virtually immediate. The important thing is to check that

ee (2) $$\dim(U^\perp) = n - \dim(U).$$

Let $\{u_1, \dots, u_m\}$ be a basis for U; extend it to a basis $\{u_1, \dots, u_n\}$ for V and let A be the matrix for β with respect to this basis. Now $U^\perp$ is equal to the orthgonal complement **with respect to the usual dot product** to the space $\langle Au_1, \dots, Au_m \rangle$, which (since A is invertible) is a space of dimension m. Now the result follows from (E65) or, equivalently, we can think of U as the null-space of the $m \times n$ matrix whose colums are $Au_1, \dots, Au_m$ and the rank-nullity theorem gives the result.

(E64) Prove that $\lambda \in k \mid kk^\sigma = 1\} = \{\epsilon/\epsilon^\sigma \mid \epsilon \in k\}$.

Answer. See p.30 of Cameron's "*Classical groups*".

(E65) Prove that the following map is a duality.

$$\langle x_1, \dots, x_\ell \rangle \longleftrightarrow [x_1, \dots, x_\ell].$$

Answer. Think of $x_1, \dots, x_\ell$ as column vectors and consider the $\ell \times n$ matrix obtained by writing them side-by-side. The rank-nullity theorem asserts that the nullity is n minus the rank. And, since the nullity equals $\dim([x_1, \dots, x_\ell]$, while the rank equals $\dim(\langle x_1, \dots, x_\ell \rangle)$, the result follows.

(E69) Show that the quadratic form Q in the lecture notes has the given matrix form.

Answer. This is Lemma 1.3 of
`http://www.math.ist.utl.pt/ ggranja/manuel.pdf`

(E71) Show that the quadratic form Q in the lecture notes polarizes to β.
(E74) Prove that if $\mathrm{char}(k) = 2$, k is perfect, and $Q : V \to k$ is non-degenerate, then $\dim(\mathrm{Rad}(\beta_Q)) \leq 1$.

Answer. Let $R = \mathrm{Rad}(\beta_Q))$. Since β_Q is identically zero on R we obtain that

$$Q(x+y) = Q(x) + Q(y),$$
$$Q(\lambda x) = \lambda^2 Q(x).$$

Since k is perfect, the map $\lambda \mapsto \lambda^2$ is a field automorphism and we conclude that Q is semilinear. Thus the kernel of Q restricted to R is a hyperplane of R. But the kernel of Q is trivial, thus $\dim(R) \leq 1$ as required.

(E76) Complete the proof of Theorem 34 in lectures.

> **Answer.** We know that $\dim(V_1) = n - \dim(W_1)$ so, to show that $V = V_1 \oplus W_1$, it is enough to show that $V_1 \cap W_1 = \{0\}$. But, since W_1 is a hyperbolic line, it is clear that $W_1^\perp \cap W_1 = \{0\}$ and we are done.
>
> Suppose that $\beta|_{V_1}$ is degenerate. Then there exists $x \in V_1$ such that $\beta(x, y) = 0$ for all $y \in V_1$. But, since $\beta(x, y) = 0$ for all $y \in W_1$, we conclude that $\beta(x, y) = 0$ for all $y \in V$, a contradiction. The same argument works in the non-singular case.

(E78) Let U_1 and U_2 be subspaces of a vector space V having the same dimension. Show that there is a subspace W of V which is a complement for both U_1 and U_2.

> **Answer.** Let $m = \dim(U_1) = \dim(U_2)$ and $n = \dim(V)$. Proceed by induction on $n - m$. If $n - m = 0$, then the result is trivial. Choose $x \in V \backslash (U_1 \cup U_2)$. Let $U_i^* = \langle U_i, x\rangle$ for $i = 1, 2$. By induction there is a subspace W that is a complement for both U_1^* and U_2^*. But now $\langle W, x\rangle$ is a complement for both U_1 and U_2. We are done.

(E80) Let (V, κ) be a formed space. Then the Witt index and the type of a maximal anisotropic subspace are determined.

> **Answer.** This is clear if V is anisotropic. Otherwise V contains hyperbolic planes. If U_1, U_2 are such, then they are isometric, so by Witt's Lemma there exists g, an isometry of V, with $U_1 g = U_2$. (Note that $U_1 \cap \mathrm{Rad}(V) = 0 = U_2 \cap \mathrm{Rad}(V)$.) Then $U_1^\perp g = U_2^\perp$. The result follows by induction.

(E81) Let (V, κ) be a formed space. Any maximal totally isotropic/ totally singular subspaces in V have the same dimension. This dimension is equal to the Witt index.

> **Answer.** Let U_1 and U_2 be such and suppose that $\dim(U_1) < \dim(U_2)$. Then any linear injection $h : U_1 \to U_2$ is an isometry. By Witt's Lemma, we can extend h to an isometry g on V. Then $U_2 g^{-1}$ is totally isotropic/ totally singular since g is an isometry. But U_1 is a proper subset of $U_2 g^{-1}$ which is a contradiction.