# INTRODUCTION TO EXPANDERS

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The aim of this lecture is to give an introduction to expander graphs, ready for the next lecture when we discuss the Bourgain-Gamburd construction of expanders using growth results [BG08]. Nearly all of this material can be found in the (excellent) undergraduate text by Davidoff, Sarnak, and Valette [DSV03]. Lubotzky also has a fine text in this area [Lub94].

This write-up will be a little more brief than previous ones; I'm a little pressed for time...

## 1. Definitions

Let $X = (V, E)$ be a graph; $V$ is the set of vertices, and $E$ is the set of edges. For us a graph always satisfies the following properties:

- $|V|$ and $|E|$ are finite;
- edges are undirected;
- there are no loops;
- the graph is simple, i.e. there is at most one edge between any two vertices.

The graph is called *$k$-regular* (for some $k \in \mathbb{Z}^+$) if every vertex has exactly $k$ neighbours.

For $F$ a subset of $V$ define $\delta F$ to be the set of edges connecting $F$ to $V \backslash F$. The *expanding* or *isoperimetric* constant of $X$ is defined to be

$$h(X) = \min \left\{ \frac{|\delta F|}{|F|} \mid F \subset V, |F| \leq \frac{1}{2}|V| \right\}.$$

We can think of $h(X)$ as a measure of how quickly information can propagate through a network. The value of $h(X)$ gives us an idea of "the worst possible bottle-neck."

We are now able to define what we mean by a family of expanders: Fix $k \geq 2$. Let $(X_m)_{m \in \mathbb{Z}^+} = (V_m, E_m)$ be a family of $k$-regular graphs. Then $(X_m)_{m \in \mathbb{Z}^+}$ is a *family of expanders* if

- $|V|_m \to \infty$ as $m \to \infty$;
- there exists $\epsilon > 0$ such that $h(X_m) > \epsilon$ for all $m \geq 1$.

## 2. An alternative formulation

We can define a family of expanders in terms of the adjacency matrix of $X$; this often turns out to be an easier thing to get one's hands on. Throughout this section we require that $X$ is $k$-regular.

Let $n = |V|$ and number the vertices of $V$ from 1 to $n$. We define $A$ to be the *adjacency matrix of* $X$: $A$ is an $n \times n$ matrix with rows and columns indexed by vertices of $X$ such that

$$A = (A_{ij}) = \begin{cases} 1, & e_{ij} \in E; \\ 0, & e_{ij} \notin E. \end{cases}$$

Observe that $A$ is an real $n \times n$ symmetric matrix. Basic linear algebra tells us that $A$ has $n$ real eigenvalues which we write as

$$\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}.$$

(Note that we are allowing the possibility of repeats.) The list $\lambda_0, \ldots, \lambda_n$ is known as the *spectrum* of $X$.

We record a series of basic results concerning the spectrum of $X$; a proof can be found in [DSV03, §1].

**Proposition 2.1.** (a) $\lambda_0 = k$;
(b) $|\lambda_i| \leq k$ *for* $1 \leq i \leq n-1$;
(c) $\lambda_0$ *has multiplicity* 1 *if and only if* $X$ *is connected;*
(d) *Suppose* $X$ *is connected. Then*
- $X$ *is bipartite if and only if* $\lambda_{n-1} = -k$;
- $\lambda_{n-2} > -k$.

Now observe that for $X$ finite, connected, and $k$-regular we have that

$$k = \lambda_0 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-2} \geq \lambda_{n-1} \geq -k.$$

An eigenvalue $\lambda$ such that $|\lambda| = k$ is called *trivial*; Prop. 2.1 implies that there are either one or two trivial eigenvalues in the spectrum (remember that we are allowing for repeats). We define $\lambda_0 - \lambda_1 = k - \lambda_1$ to be the *spectral gap* of $X$.

Now a basic result from this area gives bounds on the expanding constant in terms of the spectral gap. This result does not use any very highfalutin mathematics, but it is highly ingenious (see [DSV03, Thm. 1.2.3]).

**Theorem 1.** $\frac{k-\lambda_1}{2} \leq h(X) \leq \sqrt{2k(k-\lambda_1)}$.

Thm. 1 implies an alternative definition of an expander graph: Let $(X_m)_{m\in\mathbb{Z}^+} = (V_m, E_m)$ be a family of $k$-regular graphs. Then $(X_m)_{m\in\mathbb{Z}^+}$ is a *family of expanders* if

- $|V|_m \to \infty$ as $m \to \infty$;
- there exists $\epsilon > 0$ such that $(k - \lambda_1)(X_m) > \epsilon$ for all $m \geq 1$.

Note that the bigger the spectral gap, the better the "quality" of the expander. It turns out that we can optimise this quality, at which point the family of expanders is known as a *Ramanujan family of graphs.* More details in [DSV03, §1].

## 3. Girth

Let $X$ be a connected graph throughout this section. We define the *girth* of $X$, $g(X)$, to be the length of the shortest circuit in $X$. If $X$ has no circuits (i.e. $X$ is a *tree*), then write $g(X) = \infty$.

It is interesting to think how one might calculate the girth of a graph $X$ from its adjacency matrix $A$. In general one can't just read the girth of the graph off $A$, however we note an important related quantity: let $W_l$ be the number of walks of length $l$ starting at a vertex $i$ and ending at $i$. It is easy to see that $W_l = (A^l)_{ii}$.

Write $\chi(X)$ for the *chromatic number* of $X$. This is the smallest number of colours that we can use to colour $V$ so that no adjacent vertices have the same colour.

We mention a long-standing problem in graph theory: given positive integers $C$ and $D$, construct $X$ so that

$$\chi(X) > C, \text{ and } g(X) > D. \tag{3.1}$$

Note that $k$-regular graphs satisfy the former property for $k \geq C - 1$. However it is unclear, a priori, that a graph satisfying both the conditions of (3.1) should exist; Erdös solved this problem by using a probabilistic method to show that such graphs do indeed exist. We will use a similar result specific to Cayley graphs in the next lecture.

## 4. Cayley graphs

For $A$ a finite subset of a finite group $G$, define the *Cayley graph* $\mathcal{C}(G, A) = (V, E)$ of $G$ where

- $V$ is equal to the set of elements of $G$;
- there is an edge from a vertex $x \in G$ to a vertex $y \in G$ if and only if $y = xa$ for some $a \in A$.

Note that, strictly speaking, the definition of a Cayley graph just given does not result in a graph in the sense we have used in this lecture (for instance the Cayley graph could be directed, or have loops). We will not need this level of generality; indeed the following result gives conditions under which a Cayley graph is a graph in the sense that we understand it (along with several other useful facts).

**Proposition 4.1.** *Write* $k = |A|$.

(a) $\mathcal{C}(G, A)$ *is simple and* $k$*-regular.*
(b) $\mathcal{C}(G, A)$ *has no loop if and only if* $1 \in A$.
(c) $\mathcal{C}(G, A)$ *is undirected if and only if* $A = A^{-1}$.
(d) $\mathcal{C}(G, A)$ *is connected if and only if* $\langle A \rangle = G$.
(e) $\mathcal{C}(G, A)$ *is vertex-transitive.*

*Proof.* Easy-peasy! $\square$

## 5. The question at hand

We end with the question addressed by Bourgain and Gamburd in their paper [BG08]. Take $A$ a finite set in $SL_2(\mathbb{Z})$. Define $A_p$ a subset of $SL_2(\mathbb{Z}/p\mathbb{Z})$ to be the set we obtain by reducing all entries in elements of $A$ modulo $p$. Here's the big question:

**Is $\mathcal{C}(S_2(\mathbb{Z}/p\mathbb{Z}), A_p)$ a family of expanders?**

**Theorem 2.** *If* $|SL_2(\mathbb{Z}) : A| < \infty$*, then the answer is "yes".*

The given theorem (which is a consequence of work of Selberg; see the introduction of [BG08], as well as [Lub94, Thm. 4.3.2]) gives a partial answer to the given question. Bourgain and Gamburd's achievement is to give a precise characterization of those sets $A$ for which the answer is "yes". Their proof is particularly impressive because it also constitutes the first time that expanders have been constructed using results concerning growth in groups.

## References


[BG08] J. Bourgain and A. Gamburd, *Uniform expansion bounds for Cayley graphs of $sl_2(\mathbb{F}_p)$*, Annals of Math. **167** (2008), 625–642.

[DSV03] A. Davidoff, P. Sarnak, and A. Valette, *Elementary number theory, group theory, and Ramanujan graphs*, London Math. Soc. Student Texts, no. 55, Cambridge University Press, 2003.

[Lub94] A. Lubotzky, *Discrete groups, expanding graphs, and invariant measures*, Progress in math., vol. 195, Birkhauser, 1994.