The Finite Simple Groups I: Description

Nick Gill (OU)

November 14, 2012

Simple groups

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In this talk we will be interested in the *finite* simple groups. The study of infinite simple groups is an entirely different proposition: much wilder and much less understood.

The Classification of Finite Simple Groups

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The four families are pair-wise disjoint except for (2) and (3). We will spend the rest of the lecture discussing their various properties.

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G is a finite abelian simple group if and only if $G \cong C_p$, a cyclic group of prime order.

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We say that G is *k-transitive* if

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k-transitive groups have a tendency towards simplicity...

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1. **(5-trans)**: M_{24} *and* M_{12} *(twice);*
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So far the only proof that these really are *all* the almost simple 3-transitive groups relies on CFSG. Note that M_{12} (and M_{11} in its 4-transitive incarnation) are *sharply k-transitive*.

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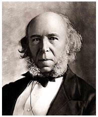

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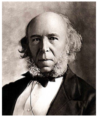

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A ces Mémoires de Cauchy j'ai toutefois emprunté une idée, et une seule : c'est celle de distinguer les fonctions en fonctions transitives et en fonctions intransitives. En effet, dans cette théorie, ce sont les fonctions transitives, et surtout celles qui le sont plusieurs fois, qui sont seules vraiment remarquables.

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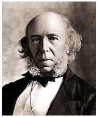

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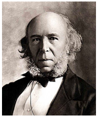

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> *...if no expert was able to fill in the details of my claims made twelve years ago, I'd better do it myself.*

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Theorem

There exist unique SS $S(5, 6, 12)$ and $S(5, 8, 24)$ such that

$$\mathrm{Aut}(S(5, 6, 12)) = M_{12} \textit{ and } \mathrm{Aut}(S(5, 8, 24)) = M_{24}.$$

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Given a SS $S(k,m,n)$ and an element $\alpha \in \Omega$, one obtains another SS $S(k-1,m-1,n-1)$ on $\Omega \backslash \{\alpha\}$ by restricting to the subset of $S(k,m,n)$ containing the element α.

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This is proved using *Iwasawa's criterion* and the fact that the natural action of $PSL_n(q)$ on 1-spaces is 2-transitive.

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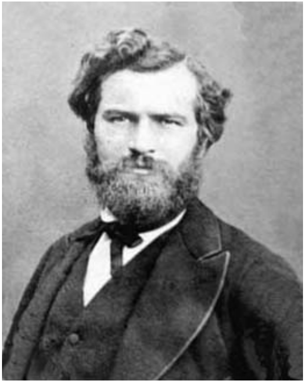

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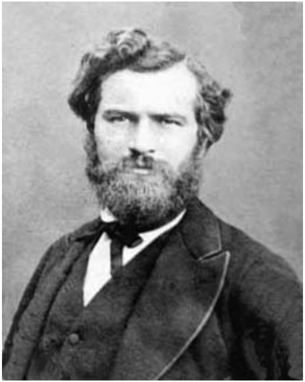

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The group G_ϕ, as well as some of its subgroups and their quotients, is known as a *finite classical group*.

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The remaining classical groups – $\mathrm{PSU}_n(q)$ (*unitary*) and $\mathrm{P}\Omega_n^\epsilon(2^a)$ (*orthogonal* with q even) – are obtained by studying isometries of sesquilinear and quadratic forms.

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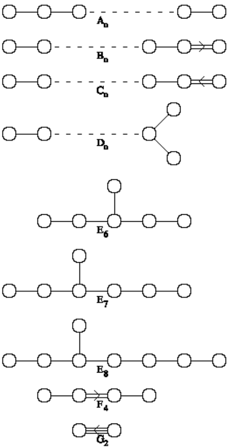

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These groups are called the *Chevalley groups*. They include most of the finite classical groups, plus some more - the first *exceptional groups*:

$$\mathrm{G}_2(q), \mathrm{F}_4(q), \mathrm{E}_6(q), \mathrm{E}_7(q), \mathrm{E}_8(q).$$

Exceptional and twisted I: Steinberg groups

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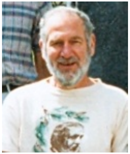

In 1959 Robert Steinberg realised that the 'missing' classical groups could be constructed by *twisting* Chevalley groups using a graph automorphism.

Exceptional and twisted I: Steinberg groups

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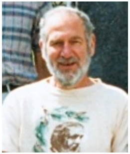

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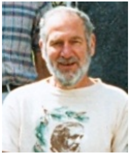

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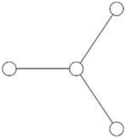

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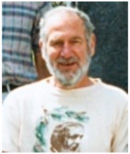

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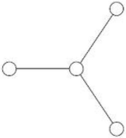

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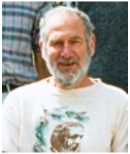

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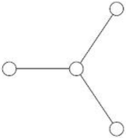

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These are the *Steinberg groups*. They include the missing finite classical groups, plus some more *exceptional groups*:

$$^3\mathrm{D}_4(q), {}^2\mathrm{E}_6(q).$$

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He obtained the *Ree groups*. They include the Suzuki groups, as well as two new families of *exceptional groups*:

$$^2\mathrm{B}_2(2^{2a+1}) = \mathrm{Sz}(2^{2a+1}), {}^2\mathrm{F}_4(2^{2a+1}), {}^2\mathrm{G}_2(3^{2a+1}).$$

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If $a > 1$ these groups are all simple. In 1964 Tits showed that $^2\mathrm{F}_4(2)$ contains a new simple group as a subgroup of index 2. This is the last simple group of Lie type.

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But that is another story...