The Finite Simple Groups II: Proof of the classification

Nick Gill (OU)

November 15, 2012

A brief recap...

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Definition

A group G is called simple if it has no non-trivial proper normal subgroups.

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4. *the 26 sporadic groups.*

Building groups

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A **composition series** *of a group G is a subnormal series*

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The quotients H_{i+1}/H_i are called the **composition factors** of G.

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Let G be a group with two composition series:

$$\{1\} = G_0 \lhd G_1 \lhd G_2 \cdots \lhd G_m = G;$$

$$\{1\} = H_0 \lhd H_1 \lhd H_2 \cdots \lhd H_n = G.$$

Then the two multisets of composition factors are equal:

$$\{G_1/G_0, \dots G_m/G_{m-1}\} = \{H_1/H_0, \dots, H_n/H_{n-1}\}.$$

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If the composition factors are all cyclic of prime order, then we call G **soluble** or **solvable** (after Galois).

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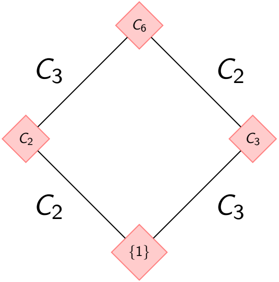

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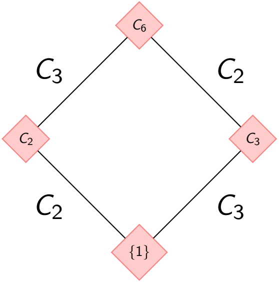

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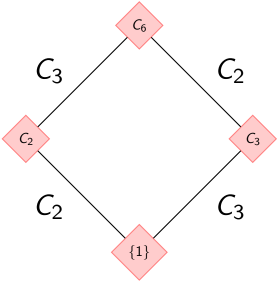

Clearly the multiset of composition factors of C_6 is $\{C_2, C_3\}$.
Warning: This is also the multiset of composition factors for $\mathrm{Sym}(3)$...

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Theorem

Let N be an abelian group, and let Q be any group. Equivalence classes of extensions

$$\{1\} \to N \to G \to Q \to \{1\}$$

are in 1-1 correspondence with the cohomology group $H^2(Q, N)$.

The proof of CFSG

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An equivalent formulation is the following:

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If G is a finite group of odd order, then G is soluble.

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3. Study the **internal structure** of G, i.e. use group theory to analyse the possible structure of the maximal subgroups of G.
4. Derive a **contradiction** somehow(!) In this case Feit and Thompson used the Brauer-Suzuki theory of exceptional characters. This theory connects the characters of a group G to the characters of its maximal subgroups.

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3. A representation is called **irreducible** if the matrices representing G do not preserve any non-trivial proper subspace of V.
4. Fact: the irreducible representations (and hence the irreducible characters) of G are in 1-1 correspondence with the conjugacy classes of G.

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χ_2	3	-1	0	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$
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Character theory would play a significant role in many of the results leading to CFSG.

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4. What is more these normalizers are Frobenius groups, i.e. they have a very special, very well understood structure.
5. One can work out the characters of G by *inducing* from the characters of these Frobenius groups.

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- Definition: A **local subgroup** of a group G is a subgroup which is equal to the normalizer of a p-group.
- It turns out that the local subgroups are the most important when it comes to understanding 'internal structure'.
- And so **local analysis** is born.

Involution centralizers

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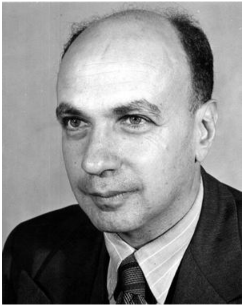

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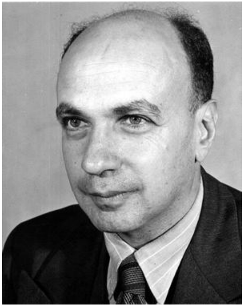

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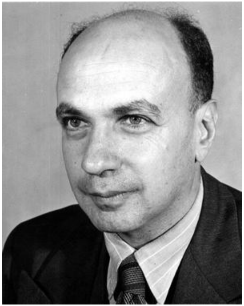

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11 of the 21 modern sporadic groups were discovered by examining involution centralizers...

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Moreover Janko gave two matrices in $GL_7(11)$ which generated this new group.

A game of two halves

The discovery of a new sporadic group often proceeded in two stages. Take the other Janko groups J_2 and J_3, for instance.

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(Janko 1969) *If G is a simple group in which the centralizer of an involution is isomorphic to $(Q_8 * Q_8).\mathrm{Alt}(5)$, then one of the following holds:*

1. *G has two classes of involutions and $|G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$;*
2. *G has one class of involutions and $|G| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$.*

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But does such a group **exist?** And, if it does, is it **unique?**

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Methods varied and, at least initially, involved the use of computers.

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Hypothesis: Suppose G is a minimal counter-example to CFSG. i.e. the composition factors of all proper subgroups of G are in one of the four families. Then...

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> *Not a single leading group theorist besides Gorenstein believed in 1972 that the Classification would be completed this century. By 1976, almost everyone believed that the Classification problem was "busted". The principal reason was Michael Aschbacher's lightning assaults on the B-Conjecture, the Thin Group Problem, and the Strongly p-embedded 2-local problem.*

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- Thompson showed that M was the only group with the given properties admitting an irreducible representation of degree 196 883.
- So, when Norton showed that all groups of this kind have such an irreducible representation, the uniqueness of M was proved.... and so was CFSG.

Some moonshine to finish

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- Suppose that f is a meromorphic function on $\mathbb{H}$.
- f is called a **modular function** if it satisfies

$$f\left(\frac{az+b}{cz+d}\right) = f(z)$$

for all transformations in $\Gamma = PSL_2(\mathbb{Z})$. We can think of f as a function on the Riemann surface $\mathbb{H}/\Gamma$.

More moonshine

- A particularly important modular function is the **elliptic modular function** j. It has Fourier series

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- These connections led Conway and Thompson to conjecture the existence of a particular graded module on which M acts naturally. What is more this action yields a 'natural construction' of the Fourier expansion of different modular functions.

The Bootleggers

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John Conway

John Thompson

Richard Borcherds

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Describing the experience of proving the Moonshine conjectures, Borcherds remarked:

> *I sometimes wonder if this is the feeling you get when you take certain drugs. I don't actually know, as I have not tested this theory of mine.*

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Thanks for coming!