Regular maps and simple groups

Nick Gill (OU)

July 30, 2013

Joint with M. Conder (Auckland), I. Short (OU), J. Širáň (OU).

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A **map** is a 'nice' embedding of $\mathcal{G}$ in $\mathcal{S}$.

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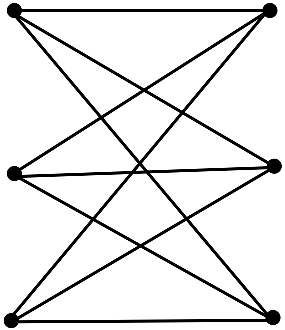

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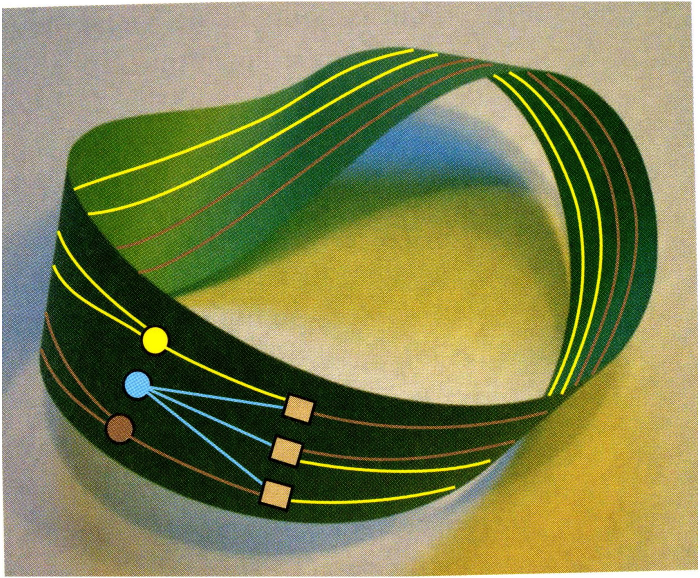

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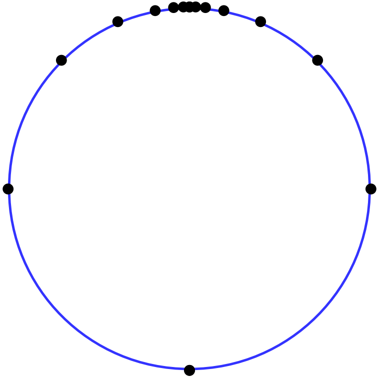

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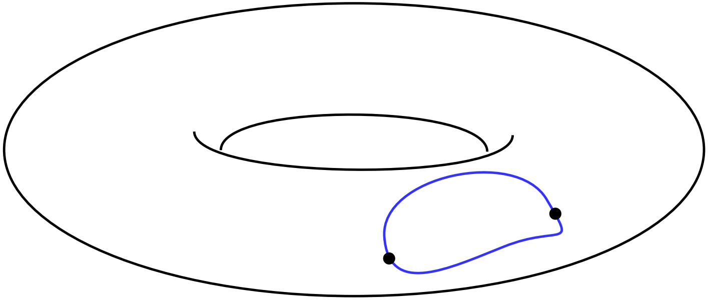

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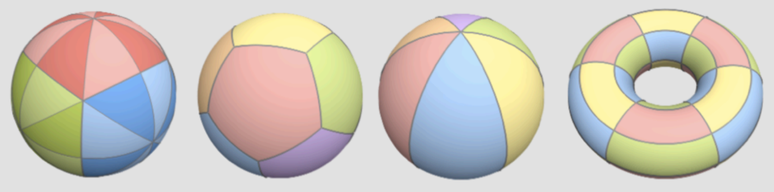

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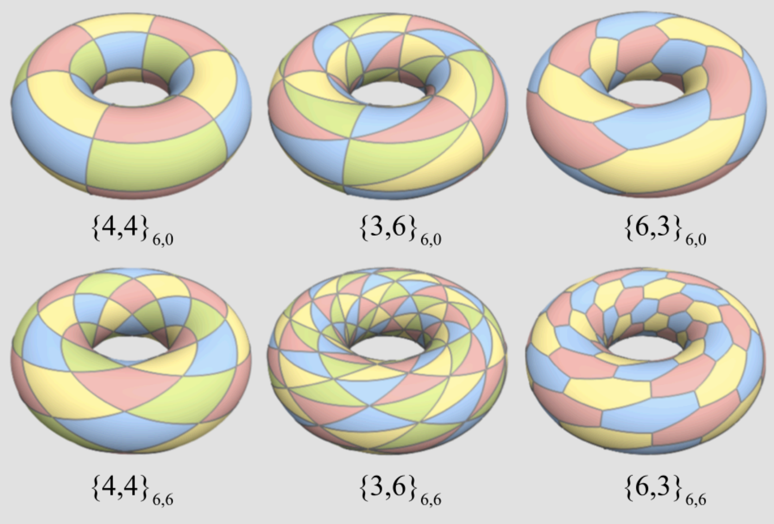

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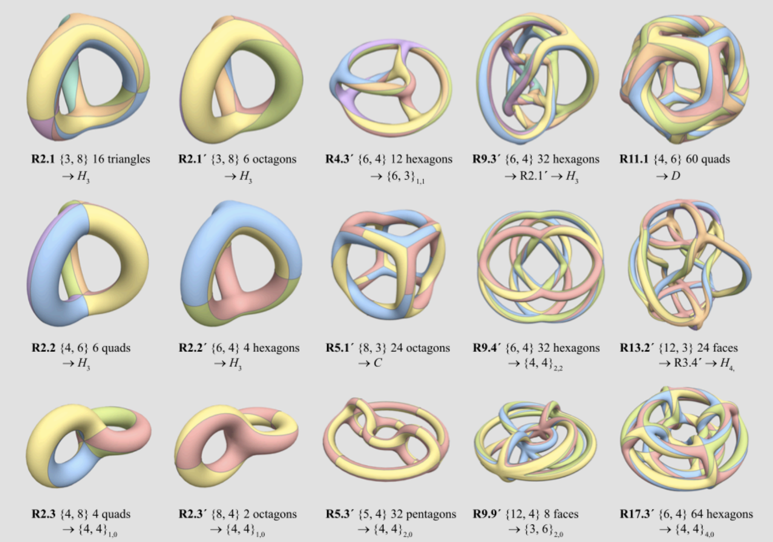

R2.1 {3, 8} 16 triangles → H_3

R2.1′ {3, 8} 6 octagons → H_3

R4.3′ {6, 4} 12 hexagons → $\{6, 3\}_{1,1}$

R9.3′ {6, 4} 32 hexagons → R2.1′ → H_3

R11.1 {4, 6} 60 quads → D

R2.2 {4, 6} 6 quads → H_3

R2.2′ {6, 4} 4 hexagons → H_3

R5.1′ {8, 3} 24 octagons → C

R9.4′ {6, 4} 32 hexagons → $\{4, 4\}_{2,2}$

R13.2′ {12, 3} 24 faces → R3.4′ → H_4,

R2.3 {4, 8} 4 quads → $\{4, 4\}_{1,0}$

R2.3′ {8, 4} 2 octagons → $\{4, 4\}_{1,0}$

R5.3′ {5, 4} 32 pentagons → $\{4, 4\}_{2,0}$

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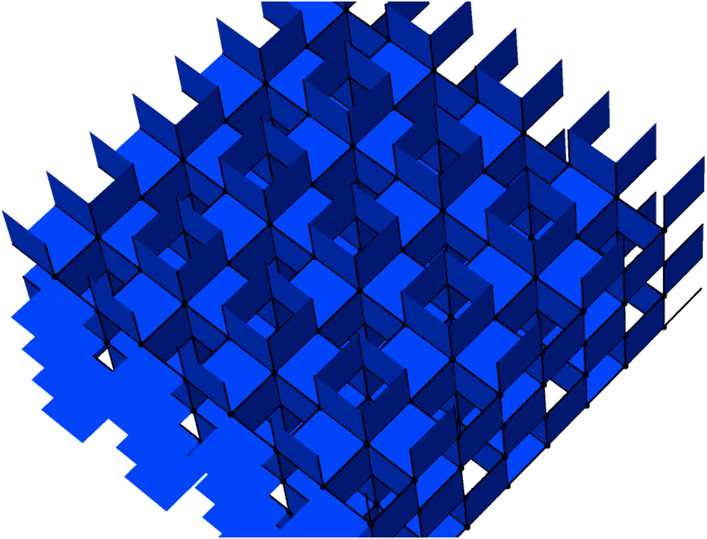
R17.3′ {6, 4} 64 hexagons → $\{4, 4\}_{4,0}$

And this one is especially groovy...



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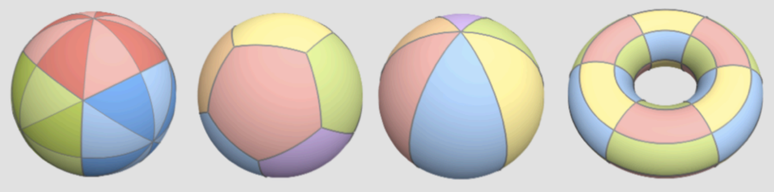
- **Fact:** $\mathrm{Aut}(\mathcal{M})$ acts faithfully and semiregularly on the set of flags.

And a flag is...



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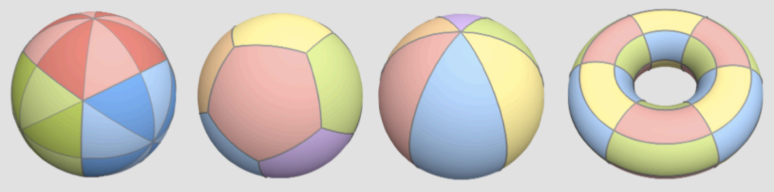

... a triple (v, e, f) where v is a vertex, e is an edge, f is a face, and all are incident with each other.

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- We would like to classify the regular maps.
- Encouraging fact: For any $g \geq 2$, there are only a finite number of regular maps on a surface of genus g.

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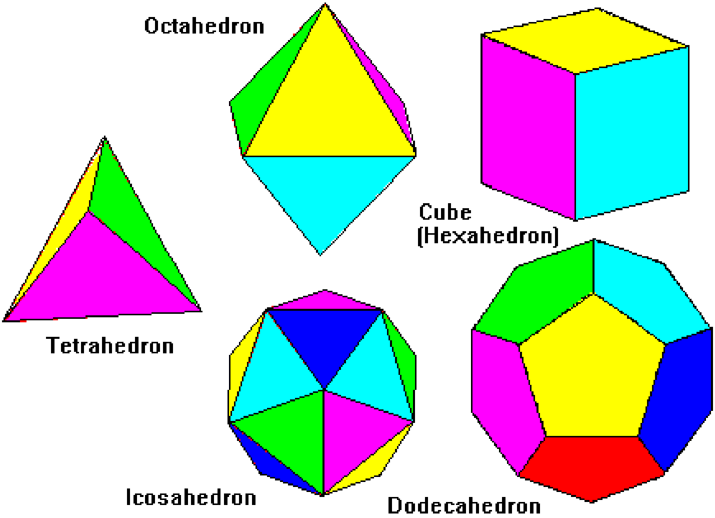

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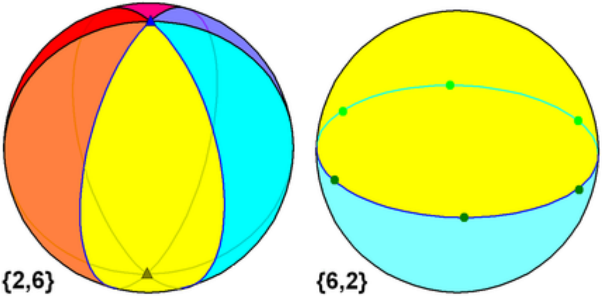

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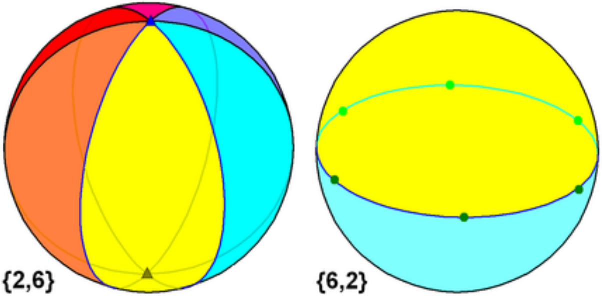

$\mathrm{Aut}(\mathcal{M})$ is solvable except when $\mathcal{M}$ is the dodecahedron or icosahedron, in which case $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Alt}(5) \times C_2$.

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- If $\mathcal{M}$ is regular with $G = \mathrm{Aut}(\mathcal{M})$, then

$$\chi = V - E + F = |G| \left(\frac{1}{|G_v|} - \frac{1}{|G_e|} + \frac{1}{|G_f|} \right).$$

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There are several infinite families of regular maps with $\chi = -p$ and solvable automorphism groups.

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Even if we drop the non-solvable condition we can still conclude that $p = 2, 3$ or 7.

A little group theory

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G is defined to be *solvable* if G has a composition series such that all quotients are abelian.

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- *One of 26 sporadic groups.*

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χ is complicated $\longleftrightarrow$ The prime factorization of χ has many primes and/or high exponents.

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Theorem (Conder, G., Short, Širáň, 2013)

$$a_1 \geq y - 2.$$

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- It is sufficient to prove that $\frac{|G|}{4[m,n]}$ is divisible by r distinct primes.

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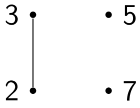

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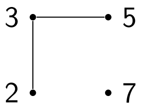

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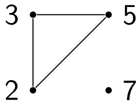

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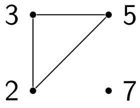
- two vertices p_i, p_j joined by an edge if and only if G contains an element of order $p_i p_j$.

$\mathrm{Alt}(9)$



Observe that if $g \in G$, then the primes dividing the order of g must all be connected in $\mathrm{Prime}(G)$.

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- We conclude that χ is divisible by at least $\sim r-2$ primes.

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$$\chi = -p^a,\ -2p^a,\ -2p_1^{a_1}p_2^{a_2}\ldots;$$

- A2 We are in the process of trying to recover the general classification of regular maps when $\chi = -p^2$, and extending it to $\chi = -p^3, -p^4, \ldots$

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This is not true in general, e.g. $\mathrm{SL}_2(5)$.

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- $F^*(G)$, the **Generalized Fitting Subgroup** of G equals

$$F(G)K_1K_2\cdots K_t$$

where $K_1, \ldots, K_t$ are the components of G.

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To understand the structure of G we need to understand the automorphisms of some quasisimple groups, and the automorphisms of some p-groups.

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In particular M is a cover of a known regular map.

Thanks for coming!