Is Babai afraid of spiders?

Nick Gill (OU)

July 4, 2011

Joint with Helfgott (ENS); Bamberg, Royle, Seress, Spiga (UWA).

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This has been proved for G a finite group of Lie type of bounded rank. The conjecture is open for groups of Lie type of unbounded rank, and for the alternating groups.

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We prove this by showing that we can write all elements of G as words (in elements of S) of length $\leq (\log |G|)^c \sim n^c$. (Stirling's formula)

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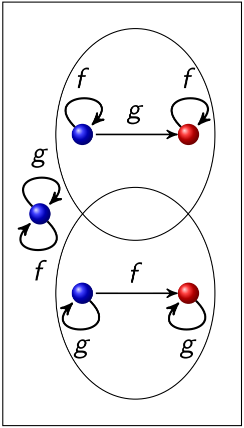

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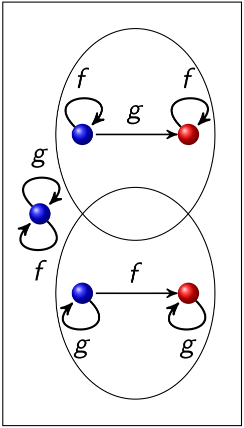

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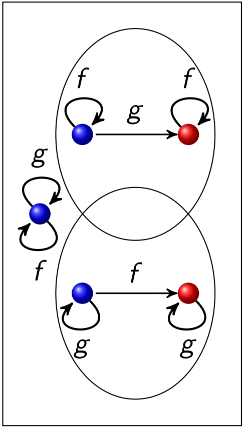

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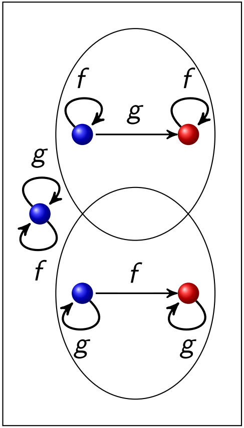

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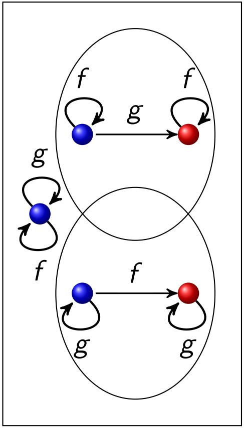

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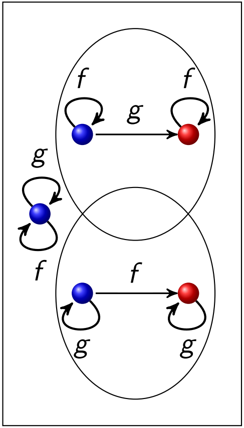

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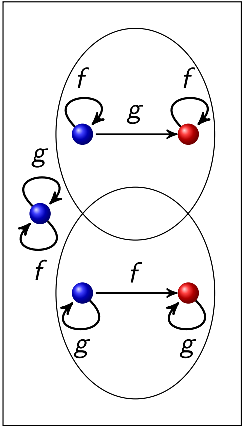

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Can we do better? Why stick with $[g, f]$? What about other words?

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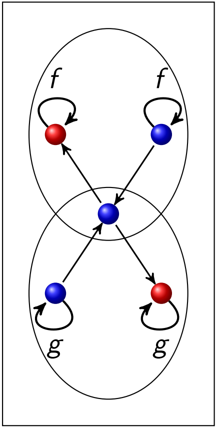

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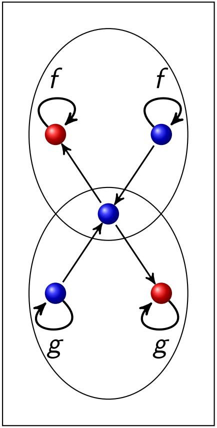

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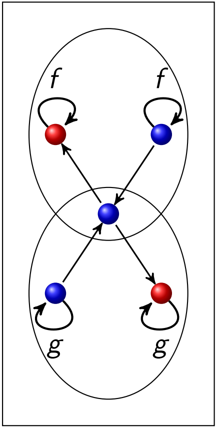

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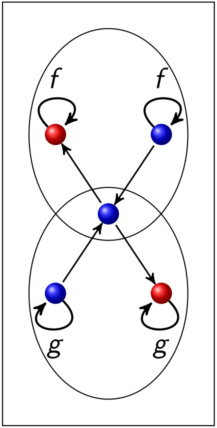

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We conclude that...

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