EXERCISE SHEET 1

Answers should be handed by Monday 19th October 2009.

1. For each of the following pairs of polynomials f and g, (i) find the quotient and remainder on dividing g by f; (ii) use the Euclidean Algorithm to find the highest common factor h of f and g; (iii) find polynomials a and b with the property that $h = af + bg$.

(a) $g = t^7 - t^3 + 5$, $f = t^3 + 7$ over $\mathbb{Q}$;
(b) $g = 4t^3 - 17t^2 + t - 3$, $f = 2t + 5$ over $\mathbb{Q}$.

2. For each of the following pairs of polynomials f and g, (i) find the quotient and remainder on dividing g by f; (ii) use the Euclidean Algorithm to find the highest common factor h of f and g; (iii) find polynomials a and b with the property that $h = af + bg$.

(a) $g = t^3 + 2t^2 - t + 3$, $f = t + 2$ over $\mathbb{F}_5$;
(b) $g = t^7 - 4t^6 - 4t + 6$, $f = 2t^3 - 2$ over $\mathbb{F}_7$.

3. A non-zero polynomial $f \in \mathbb{Z}[t]$ is *primitive* if its coefficients are relatively prime.

(a) Prove Gauss' lemma: the product of two primitive polynomials in $\mathbb{Z}[t]$ is also primitive.
(b) Let f be a polynomial in $\mathbb{Z}[t]$ which is irreducible over $\mathbb{Z}$. Prove that f, considered as a polynomial in $\mathbb{Q}[t]$, is irreducible over $\mathbb{Q}$. (Suggestion: Suppose that $f \in \mathbb{Z}[t]$ and $g, h \in \mathbb{Q}[t]$ are all monic such that $f = gh$. Show there exist $m, n \in \mathbb{Z}^+$ such that mg and nh are primitive. Then consider the equation $(mn)f = mg.n.$)

4.

(a) Show that the polynomial $t^2 + t + 2$ is irreducible in $\mathbb{F}_3[t]$.
(b) Give a complete list of the coset representatives of the quotient ring $\mathbb{F}_3[t]/(t^2+t+2)$.
(c) For each of the non-zero elements α of $\mathbb{F}_3[t]/(t^2 + t + 2)$, calculate α^{-1}.
(d) For each of the non-zero elements α of $\mathbb{F}_3[t]/(t^2+t+2)$, determine the least integer n (if one exists) for which $\alpha^n = 1$.

5. Let K be a field. A non-constant polynomial f over K is called *prime* if, whenever $f \mid gh$ either $f \mid g$ or $f \mid h$. Prove that a non-constant polynomial is prime if and only if it is irreducible. (The analogous result fails for general Unique Factorization Domains).