11. Projective Space

(E11.1) Show that $PG_2(2)$ and the Fano plane are the same incidence structure. (We would do better to write that "$PG_2(2)$ and the Fano plane are isomorphic as incidence structures", but we have not yet defined what we mean by isomorphism.)

(E11.2*) Show that, for any prime power q, $PG_2(q)$ is an abstract projective plane.

(E11.3) Prove (b).

(E11.4) Prove that $\lim_{q\to 1} \begin{bmatrix} n \\ m \end{bmatrix}_q = \binom{n}{m}$.

(E11.5) Prove that

$$\begin{bmatrix} n \\ m \end{bmatrix}_q + q^{n-m+1}\begin{bmatrix} n \\ m-1 \end{bmatrix}_q = \begin{bmatrix} n+1 \\ m \end{bmatrix}_q.$$

(E11.6*)Prove that this action is well-defined, and that the action preserves the incidence relation for $PG(V)$.

(E11.7) Prove that $\ker(\phi) = \{\alpha I \in GL(V) \mid \alpha \in k\}$.

(E11.8) Prove that K is central in $GL(V)$. Can you characterize those fields k and those vector spaces V for which K is central in $\Gamma L(V)$?

(E11.9*)Prove that

$$|PGL_n(\mathbb{R}) : PSL_n(\mathbb{R})| = \begin{cases} 1, & \text{if } n \text{ is odd;} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

(E11.10)V^* is a vector space over k of dimension n.

(E11.11)$U \mapsto U^\dagger$ is a bijection between the subspaces of V and the subspaces of V^*.

(E11.12) $U_1 \le U_2$ if and only if $U_1^\dagger \ge U_2^\dagger$.

(E11.13) If $U \le V$, then $\dim(U^\dagger) = n - \dim(U)$ and $\text{pdim}(U^\dagger) = n - 2 - \text{pdim}(U)$

(E11.14) Prove that $U \to U^\dagger$ is a weak automorphism of $PG(V)$.

(E11.15*) Prove that, for $n \ge 3$, $WAut(PG_n(q))$ contains $Aut(PG_n(q))$ as an index 2 subgroup. Can you say any more about the structure of $WAut(PG_n(q))$?

(E11.16)$PG_{n-1}(q)$ is a thick abstract projective space.