9. Series

(E9.1) Prove this. (This is hard.)

(E9.2) Prove this.

(E9.3) Prove that, for N a normal subgroup of G, the quotient G/N is abelian if and only if $G' \leq N$.

(E9.4) Find an example of a group G such that G' is not equal to the set of all commutators. (This is tricky; if you know about free groups, then I'd start there...)

(E9.5) Prove that (provided it terminates) the derived series is a normal series.

(E9.6) Prove that, if G is finite, then G is solvable if and only if all composition factors of G are cyclic of prime order. Give an example of a solvable group that does not have a composition series.

(E9.7) Prove that a finite group G is solvable if and only if the derived series of G terminates at $\{1\}$.

(E9.8) What is another name for a nilpotent group of class 1?

(E9.9) Prove that a p-group is nilpotent.

(E9.10) A group is nilpotent if and only if the lower central series terminates at $\{1\}$. The nilpotency class of a nilpotent group G is equal to the length of the lower central series.

(E9.11) Prove that, for all i, Z_{i+1}/Z_i is the center of G/Z_i. Deduce that a group is nilpotent if and only if the upper central series terminates at G. The nilpotency class of a nilpotent group G is equal to the length of the upper central series.

(E9.12) Prove that if a prime t divides the order of a finite nilpotent group G, then G has a unique Sylow t-subgroup. Deduce that G is the direct product of its Sylow subgroups.

(E9.13) Prove that if G is solvable, then $C_G(F(G)) = Z(F(G))$.

(E9.14) Use Iwasawa's criterion to show that A_5 is simple.

(E9.15) Now use Iwasawa's criterion to show that A_n is simple for $n \geq 5$. Hint: consider the action on unordered triples from $\{1, \ldots, n\}$.

(E9.16) Prove the following variant on Iwasawa's criterion: Suppose that G is a finite perfect group acting faithfully and primitively on a set Ω, and suppose that the stabilizer of a point has a normal soluble subgroup S, whose conjugates generate G. Then G is simple.