7. Minimal normal subgroups and the socle

Throughout this section $G$ is a nontrivial group. A minimal normal subgroup of $G$ is a normal subgroup $K \neq 1$ of $G$ which does not contain any other nontrivial normal subgroup of $G$. The socle, $\text{soc}(G)$ is the subgroup generated by the set of all minimal normal subgroups (if $G$ has no minimal normal subgroups, then we set $\text{soc}(G) = 1$).

(E7.1) Find the socle of $D_{10}$, $A_4$, $S_4$, $S_4 \times Z$.

(E7.2) Give an example of a group that has no minimal normal subgroup.

(E7.3) If $G$ is the direct product of a (possibly infinite) number of finite simple groups, then what is $\text{soc}(G)$?

(E7.4) Give a characterization of almost simple groups in terms of their socle.

Clearly $\text{soc}(G)$ is a characteristic subgroup of $G$.\textsuperscript{26}

**Theorem 7.1.** Let $G$ be a nontrivial finite group.

1. If $K$ is a minimal normal subgroup of $G$, and $L$ is any normal subgroup of $G$, then either $K \leq L$ or $\langle K, L \rangle = K \times L$.\textsuperscript{27}
2. There exist minimal normal subgroups $K_1, \ldots, K_m$ of $G$ such that $\text{soc}(G) = K_1 \times \cdots \times K_m$.
3. Every minimal normal subgroup $K$ of $G$ is a direct product $K = T_1 \times \cdots \times T_k$ where the $T_i$ are simple normal subgroups of $K$ which are conjugate in $G$.
4. If the subgroups $K_i$ in (2) are all nonabelian, then $K_1, \ldots, K_m$ are the only minimal normal subgroups of $G$.

**Proof.**

1. Since $K \cap L \trianglelefteq G$, the minimality of $K$ implies that either $K \leq L$ or $K \cap L = \{1\}$. In the latter case, since $K$ and $L$ are both normal, we have that $\langle K, L \rangle = KL = K \times L$.
2. Because $G$ is finite we can find a set $S = \{K_1, \ldots, K_m\}$ of minimal normal subgroups which is maximal with respect to the property that $H := \langle S \rangle = K_1 \times \cdots \times K_m$. We must show that $H$ contains all minimal normal subgroups of $G$, and so is equal to $\text{soc}(G)$. This follows immediately from (1).
3. Let $T$ be a minimal normal subgroup of $K$ and observe that all conjugates $T^g$, for $g \in G$, are also minimal normal subgroups of $K$. Choose a set $S = \{T_1, \ldots, T_m\}$ of these conjugates which is maximal with respect to the property that $L := \langle S \rangle = T_1 \times \cdots \times T_m$. Then, arguing à la (2), we see that $L$ contains all of the conjugates of $T$ under $G$ and so $L \trianglelefteq G$. But, since $\{1\} < L \leq K$ and $K$ is minimal normal, we conclude that $K = L = T_1 \times \cdots \times T_m$. Note, finally, that, for $T_i$ to be minimal normal in $T_1 \times \cdots \times T_m$, we must have $T_i$ simple.
4. Let $K$ be a minimal normal subgroup of $G$ that is distinct from $K_1, \ldots, K_m$, then, applying (1) with $L = K_i$, we find that $\langle K, L \rangle = K \times L$ and, in particular, $K$ centralizes each of the $K_i$. Thus $K \leq Z(\text{soc}(G))$. However if each $K_i$ is nonabelian, then (3) implies that $Z(K_i) = \{1\}$ and we have a contradiction, as required. \hfill $\square$

Observe that if a minimal normal subgroup $K$ is abelian, then $K$ is an elementary-abelian $p$-group for some prime $p$.\textsuperscript{28}

(E7.5) Suppose that $G$ is elementary abelian. How many minimal (resp. maximal) normal subgroups does $G$ have?

\textsuperscript{26}I.e. $\text{soc}(G)$ is invariant under any automorphism of $G$. This is because any automorphism of $G$ must permute the minimal normal subgroups of $G$.

\textsuperscript{27}I am considering the internal direct product here, a special case of the internal semidirect product that we have already seen.

\textsuperscript{28}Recall that an elementary-abelian $p$-group is defined to be a group that is isomorphic to $C_p \times \cdots \times C_p$, where $n$ is finite, $p$ is a prime, and $C_p$ is a cyclic group of order $p$. 
7.1. Finite groups with elementary abelian socle. In this section \( G \) is a finite group with an elementary-abelian socle. We write

\[
V := \text{soc}(G) = C_p \times \cdots \times C_p.
\]

Observe that \( V \) has a natural structure as a vector space over \( \mathbb{F}_p \), the field of order \( p \). This allows us to make the following assertion.

**Lemma 7.2.** \( G/C_G(V) \) is isomorphic to a subgroup of \( GL(V) \), and \( G/C_G(V) \) acts on \( V \) via multiplication (on the right) by matrices.

**Proof.** Consider the action of \( G \) on \( V \) by conjugation. Lemma 3.2 implies that this induces a homomorphism \( \phi : G \to \text{Aut}(V) \) where we view \( V \) as an object from \textbf{Group}. Furthermore \( C_G(V) \) is the kernel of \( \phi \) and, in particular, it is a normal subgroup of \( G \).

Now, since \( C_G(V) \) is the kernel of the conjugation action, the first isomorphism theorem of groups implies that \( G/V \) is isomorphic to a subgroup of \( \text{Aut}(V) \). Now the result follows from (E7.6) below. \( \square \)

(E7.6) Let \( V \) be an elementary-abelian \( p \)-group.

1. Let \( G \) be a group of automorphisms of \( V \) as an object from \textbf{Group}. Prove that \( G \) acts linearly on \( V \), i.e. prove that \( G \) is a group of automorphisms of \( V \) as an object from \textbf{Vect}_{\mathbb{F}_p} \).

2. Let \( G \) be a group of automorphisms of \( V \) as an object from \textbf{Vect}_{\mathbb{F}_p} \). Prove that \( G \) is a group of automorphisms of \( V \) as an object from \textbf{Group}.

3. Conclude that \( \text{Aut}(V) = GL(V) \), whether we consider it an object of \textbf{Group} or of \textbf{Vect}_{\mathbb{F}_p} \).

Let us strengthen our supposition: let us suppose that \( G \) splits over \( V \), i.e. that there exists a subgroup \( H < G \), such that \( G = V \rtimes H \). To state the structure result in this case, we need a definition.

Given a vector space \( V \) over a field \( K \), define

\[
\text{AGL}(V) := \{(g, v) \mid v \in K^d, g \in GL(V)\} = K^d \rtimes GL(V).
\]

Here we write \( K^d \) to mean the additive group whose elements are \( d \)-tuples with entries from \( K \). Multiplication is defined in the usual way for a semidirect product:

\[
(g_1, v_1)(g_2, v_2) := (g_1, g_2, v_1^g v_2)
\]

where, for \( v \in K^n \) and \( g \in GL_d(K) \), we define \( v^g \) to be the product of \( v \) (thought of as a row vector) with \( g \), a matrix.

Note that, just as with \( GL(V) \), we will write \( \text{AGL}_d(K) \) as a pseudonym for \( \text{AGL}(V) \). Furthermore, if \( K = \mathbb{F}_p \) is finite, then we will write \( \text{AGL}_d(p) \) as a synonym for \( \text{AGL}_d(\mathbb{F}_p) \).

**Proposition 7.3.** Suppose that \( G \) is a finite group with socle \( V \) of order \( p^d \) for some prime \( p \). If \( G \) splits over \( V \), then \( G \) is isomorphic to a subgroup of \( \text{AGL}_d(p) \).

**Proof.** Let \( H \) be a subgroup of \( G \) such that \( G = V \rtimes H \). Consider the action of \( H \) on \( V \) by conjugation.

We claim that \( C_H(V) \) is a normal subgroup of \( G \); indeed it is enough to prove that it is a normal subgroup of \( H \), since \( G = VH \) and \( C_H(V) \) is certainly normalized by \( V \). To prove the claim, take \( h \in C_H(V), h_1 \in H \) and \( v \in V \). Now observe that

\[
v^{(h_1)} = (h_1)^{-1}v(h_1) = (h_1^{-1}hh_1)^{-1}v(h_1^{-1}hh_1) = h_1^{-1}(h_1^{-1}(1h_1^{-1})h)h_1 = h_1^{-1}(h_1v)h = v.
\]

Thus \( h_1^h \in C_H(V) \) and the claim is proved. But now, since \( H \cap V = \{1\} \), we know that \( C_H(V) \cap V = \{1\} \) and so, since \( V \) is the socle, we conclude that \( C_H(V) = \{1\} \).

Now consider \( C_G(V) \). If \( g \in G \), then \( g = vh \) for a unique \( v \in V \) and \( h \in H \) and \( g \in C_G(V) \) if and only if \( h \in C_G(V) \). In particular,

\[
C_G(V) = V.C_H(V) = V.
\]
Now Lemma 7.2 implies that $G/V$ is isomorphic to a subgroup of $GL(V)$. Moreover the group $H \cong G/V$ acts on $V$ by right multiplication by matrices; in other words $V \rtimes H$ is isomorphic to a subgroup of $AGL_d(p)$ as required.

(E7.7) Suppose that $K$ is finite.
- Prove that $\text{soc}(AGL_d(K)) \cong K^d$ and, moreover, that $\text{soc}(AGL_d(K))$ is a minimal normal subgroup of $AGL_d(K)$.
- Suppose that $\text{soc}(AGL_d(K)) \leq G \leq AGL_d(K)$. Under what conditions is $\text{soc}(AGL_d(K))$ a minimal normal subgroup of $G$?

(E7.8) Describe the structure of $AGL_1(p)$ for a prime $p$.

7.2. The socle of a primitive permutation group.

Theorem 7.4. If $G$ is a finite primitive subgroup of $\text{Sym}(n)$, and $K$ is a minimal normal subgroup of $G$, then exactly one of the following holds:

1. for some prime $p$ and some integer $d$, $K$ is a regular elementary abelian group of order $p^d$, and $\text{soc}(G) = K = C_G(K)$.
2. $K$ is a regular non-abelian group, $C_G(K)$ is a minimal normal subgroup of $G$ which is permutation isomorphic to $K$, and $\text{soc}(G) = K \times C_G(K)$.
3. $K$ is non-abelian and $\text{soc}(G) = K$.

Proof. Let $C = C_G(K)$. Since $C \leq G$, either $C = 1$ or $C$ is transitive. Since $K$ is transitive, Lemma 3.7 implies that $C$ is semiregular, and hence either $C = 1$ or $C$ is regular.

Suppose that $C = 1$. Clearly $K$ is non-abelian. Furthermore Theorem 7.1 (1) implies that $K$ is the only minimal normal subgroup of $G$ and so $\text{soc}(G) = K$ and conclusion (3) of the result holds.

Suppose instead that $C$ is regular. Then (E3.23) implies that $C_{\text{Sym}(\Omega)}(C)$ is regular. Since $K \leq C_{\text{Sym}(\Omega)}(C)$ and $K$ is transitive, we conclude that $K = C_{\text{Sym}(\Omega)}(C)$. Similarly, $C = C_{\text{Sym}(\Omega)}(K)$ and, by (E3.24), $C$ and $K$ are permutation isomorphic.

Now, since $C$ is regular, we conclude that $C$ is a minimal normal subgroup of $G$ (since any proper subgroup of $C$ is intransitive). Now Theorem 7.1 (1) states that every minimal normal subgroup of $G$ distinct from $K$ is contained in $C$. Thus $\text{soc}(G) = KC$ which equals $K$ or $K \times C$ depending on whether $C \leq K$ or not.

If $C \leq K$, then $C = K$, $K$ is abelian, $\text{soc}(G) = K$ and conclusion (1) holds. If $C \nleq K$, then conclusion (2) holds.

(E7.9) Suppose that $G$ is a maximal primitive subgroup of $\text{Sym}(n)$. Prove that $G$ has a unique minimal normal subgroup (and so possibility (2) in Theorem 7.4 cannot occur).²⁹

(E7.10) Suppose that $K$ is a regular normal subgroup of $G$, a subgroup of $\text{Sym}(n)$. Let $H$ be the stabilizer of a point in the action on $\Omega := \{1, \ldots, n\}$. Then $G = KH, K \cap H = \{1\}$ and, in particular, $G$ splits over $K$, i.e. $G = K \rtimes H$.

7.3. Primitive permutation groups with abelian socle. Our approach now is similar to that of §§5 and 6. In those sections we studied subgroups $G$ of $\text{Sym}(n)$ that preserved certain structures on the set $\{1, \ldots, n\}$ (in §5, this was a $G$-congruence; in §6 it was a $G$-product structure). In this section the structure of interest is a $G$-affine structure.

It will be convenient to take a more categorical approach here, simply because the category associated to a $G$-affine structure is very standard. The category of interest is called $\textbf{Aff}_K$; it is very similar to $\textbf{Vect}_K$, but we are allowing more arrows.³⁰

Objects: Objects are finite-dimensional vector spaces over the field $K$.

Arrows: An arrow $g : V_1 \to V_2$ is an affine transformation, i.e. a map that acts linearly on the difference between two vectors. In other words $g : V_1 \to V_2$ is an affine transformation if there exists a

²⁹ You may find it helpful to refer to the proof of (E3.23) and (E3.24).
³⁰ We have done this before – see Example 3 – but in a different direction.
linear transformation $h : V_1 \rightarrow V_2$ such that
\[ v_2^g - v_1^g = (v_2 - v_1)^h. \]

Suppose that $V$ is an object in $\text{Aff}_K$. For $x \in V$ define the map
\[ n_x : V \rightarrow V, v \mapsto v + x. \]
The map, $n_x$, is called the translation by the vector $x$ and one can check that $n_x$ is an arrow in $\text{Aff}_K$.\(^{31}\)

The set of all translations
\[ N := \{ n_x \mid x \in V \} \]
is a subgroup of $\text{Aut}(V)$. It is clear that any linear transformation of $V$ is also an affine transformation, thus $\text{GL}(V)$ is also a subgroup of $\text{Aut}(V)$.

\textbf{(E7.11)} Let $V$ be a vector space. Prove that
- $N \cap \text{GL}(V) = \{1\}$, where $N$ is the set of translations of $V$;
- $\text{Aut}(V) = N \rtimes \text{GL}(V) \cong \text{AGL}_d(K)$ where $d = \dim(V)$, the dimension of $V$ as a vector space over $K$;
- $\text{AGL}(V)$ acts faithfully and 2-transitively on $V$;
- The stabilizer of the zero vector is $\text{GL}(V)$.

Now let $G$ be a subgroup of $\text{Sym}(\Omega)$ where $\Omega$ is a finite set. An affine structure is a bijection $\theta : \Omega \rightarrow V$, where $V$ is a finite-dimensional vector space over a finite field $K$. An affine structure is a $G$-affine structure if $G$ acts on $V$ as an object from $\text{Aff}_K$.\(^{32}\) By Lemma 3.2, if a $G$-affine structure exists, then the action of $G$ on $\Omega$ yields a homomorphism $G \rightarrow \text{Aut}(V)$. By (E7.11) $\text{Aut}(V) \cong \text{AGL}_d(p)$ for some positive integer $d$ and some prime $p$.

\textbf{Lemma 7.5.} Let $\Omega$ be a finite set of order $n$. Suppose that $\theta : \Omega \rightarrow V$ is an affine structure, where $V$ is a $d$-dimensional vector space over a finite field $K = \mathbb{F}_p$.

1. $\theta$ is a $G$-affine structure for a unique subgroup $G$ of $\text{Sym}(\Omega)$ that is isomorphic to $\text{AGL}_d(p)$;
2. if $\theta$ is a $H$-affine structure for some group $H \leq \text{Sym}(\Omega)$, then $H \leq G$.

\textbf{Proof.} We use $\theta$ to identify $\Omega$ with $V$ throughout. We have seen that $\text{Aut}(V) \cong \text{AGL}_d(p)$ where we consider $V$ an object from $\text{Aff}_K$. By (E7.11), the action of $\text{AGL}(V)$ on $V$, as a set, is faithful, thus $\text{AGL}(V)$ is a subgroup of $\text{Sym}(V)$.

Now suppose that $\theta$ is a $H$-affine structure for some group $H \leq \text{Sym}(\Omega)$. By definition every element of $h$ is an arrow in $\text{Aff}_K$ and so lies in $\text{AGL}(V)$. We are done. \(\square\)

Lemma 7.5 can be combined with Proposition 6.5 to yield the following.

\textbf{Proposition 7.6.} Let $H \leq \text{Sym}(\Omega)$ where $|\Omega| < \infty$. One of the following holds:

1. $H$ is intransitive and $H \leq \text{Sym}(k) \times \text{Sym}(n - k)$ for some $1 < k < n$;
2. $H$ is transitive and imprimitive and $H \leq \text{Sym}(k) \rtimes \text{Sym}(\ell)$ for some $1 < k, \ell < n$ with $n = kl$;
3. $H$ is primitive, preserves a non-trivial product structure, and $H \leq \text{Sym}(k) \rtimes \text{Sym}(\ell)$ for some $1 < \ell < n, 2 < k < n$ with $n = kl$;
4. $H$ is primitive, preserves an affine structure, and $H \leq \text{AGL}_d(p)$ for some $d > 1$ and prime $p$ with $n = p^d$;
5. $H$ is primitive and preserves neither an affine structure nor a non-trivial product structure.

We have seen examples of subgroups of $\text{Sym}(\Omega)$ of every given type, except those that are primitive and preserve an affine structure. To see that the latter type of group really exists, recall that $\text{AGL}_d(p)$ acts 2-transitively on the associated vector space, thus, in particular, it is primitive; of course, by definition, it also preserves an affine structure.

\(^{31}\)Indeed, a translation is precisely an affine map for which the associated linear map is the identity.

\(^{32}\)As usual, I am identifying $\Omega$ with $\theta(\Omega)$ so that I can talk of ‘$G$ acting on $V$’. If I wanted to make this precise, I would define the action of $G$ on $V$ via
\[ v^g := \theta((\theta^{-1}(v))^g). \]
\((E7.12)\) Let \(AGL_d(p) = \text{Aut}(\Omega)\) for \(\Omega\) an object from \(\text{Aff}_K\). Can you specify necessary and sufficient conditions for a subgroup \(G \leq \text{Aut}(\Omega)\) to act primitively on \(\Omega\).

The following exercise implies that Proposition 7.6 can be strengthened by requiring that \(4 < k\) for possibility (3).

\((E7.13)\) Let \(G\) be a subgroup of \(\text{Sym}(k^f)\) with \(k \in \{3,4\}\), and suppose that \(G \cong \text{Sym}(k) \wr \text{Sym}(\ell)\) in the product action. Prove that if \(k \leq 5\), then \(G\) preserves an affine structure, and describe the group \(AGL_d(p)\) in \(\text{Sym}(k^f)\) that contains \(G\).

### 7.4. The socle and affine structures

Our job now is to connect our knowledge about the socle of a primitive permutation group with the work in the previous section on affine structures.

**Lemma 7.7.** Suppose that \(G\) is a subgroup of \(\text{Sym}(\Omega)\) and that \(G\) contains a normal regular subgroup \(K\). Let \(H\) be the stabilizer of any point of \(\omega\). The action of \(G\) on \(\Omega\) is permutation isomorphic to the action of \(G\) on the set \(K\) given by

\[
\varphi : G \times K \rightarrow K, (g, k_1) \mapsto (k_1 k)^h
\]

where \(g = kh\) for some \(k \in K\) and \(h \in H\).

It is important to realise that the given action of \(G\) on \(K\) is not an action of \(G\) on \(K\) as an object from Group.\(^{33}\) Note, too, that we are not assuming that \(\Omega\) is finite here.

**Proof.** Fix \(\omega \in \Omega\) and define a function \(\beta : K \rightarrow \Omega, k \mapsto \omega^k\). Since \(K\) is regular this function is a bijection. Let \(1 : G \rightarrow G\) be the identity map, and let \(\psi\) be the action of \(G\) on \(\Omega\) given by the embedding of \(G\) in \(\text{Sym}(\Omega)\). Now the result is equivalent to proving that the following diagram commutes:

\[
\begin{array}{ccc}
G \times K & \xrightarrow{\varphi} & K \\
(1, \beta_\omega) \downarrow & & \downarrow (1, \beta_\omega) \\
G \times \Omega & \xrightarrow{\psi} & \Omega \\
& & \downarrow \beta_\omega \\
& & (g, \omega^{k_1}) \xleftarrow{\psi} (\ast)
\end{array}
\]

Write \(g = kh\) for some \(k \in K\) and \(h \in H\). If we follow the diagram from the top-left corner, down and across, then \((\ast) = \omega^{k_1g} = \omega^{k_1kh}\). On the other hand, if we go right and then down, we obtain

\[
(\ast) = \omega^{(k_1k)^h} = \omega^{h^{-1}k_1kh} = (\omega^{h^{-1}})^{k_1kh} = \omega^{k_1kh}
\]

and we are done. \(\square\)

**Proposition 7.8.** Suppose that \(G\) is a subgroup of \(\text{Sym}(n)\) and that \(G\) contains a normal regular elementary abelian subgroup \(V\). Then there is a \(G\)-affine structure \(\theta : \Omega \rightarrow V\), and there is a group \(M\) such that \(G \leq M \leq \text{Sym}(n)\) with \(M \cong AGL(V)\).

**Proof.** By Lemma 7.7 we know that the action of \(G\) on \(\Omega\) is permutation isomorphic to the action \(\varphi\) on \(V\). Thus it is enough to show that \(\varphi\) is an action of \(G\) on \(V\) as an object from \(\text{Aff}_K\).

Let \(H\) be the stabilizer of 0, the identity element in \(V\), and recall that, by (E7.11), \(H = \text{GL}(V)\). Let \(g \in G\) and let \(h \in H, v \in V\) be the unique elements such that \(g = vh\). Let \(v_1, v_2 \in V\) and observe that

\[
v_1^g - v_2^g = (v_1 + v)^h - (v_2 + v)^h = v_1^h + v_2^h - v_2^h - v^h = v_1^h - v_2^h = (v_1 - v_2)^h
\]

as required. \(\square\)

\(^{33}\)To see why you should think of the effect of this action on the identity of \(K\).
7.5. **What is left.** Recall that we are trying to understand the subgroups of \( \text{Sym}(n) \), and that, by Proposition 5.8, this amounts to understanding the primitive subgroups. Now Theorem 7.4 gives three possibilities for a primitive subgroup, and Proposition 7.8 gives a full description for possibility (1).

If, moreover, we restrict our attention to *maximal* primitive subgroups of \( \text{Sym}(n) \), then (E7.9) implies that possibility (2) cannot occur.

Thus we can state the following corollary to Theorem 7.4:

**Corollary 7.9.** If \( M \) is a finite maximal primitive subgroup of \( \text{Sym}(n) \), and \( K \) is a minimal normal subgroup of \( G \), then exactly one of the following holds:

1. \( n = p^d \) for some prime \( p \) and some integer \( d \), and \( M = \text{AGL}_d(p) \).
2. \( K \) is non-abelian and \( \text{soc}(G) = K \).

Thus, to understand the maximal subgroups of \( \text{Sym}(n) \), we need to understand those finite primitive groups that have a unique minimal normal subgroup \( K \) that is nonabelian. It is beyond the scope of this course to properly analyse this situation, although we will at least be able to state a theorem pertaining to this situation in the next section. Before we get there, though, let us observe that we have already seen two examples of primitive subgroups of this kind:

- In Exercise 16 we looked at actions of \( \text{Sym}(n) \) on the coset spaces of maximal subgroups other than \( \text{Alt}(n) \). This situation can be generalized to cover the action of any almost simple group \( G \) on the coset space of a maximal subgroup that does not contain \( \text{soc}(G) \).
- In §6 we considered the product action of the wreath product \( W := \text{Sym}(k) \wr \text{Sym}(\ell) \), and the next exercise shows we have another example.

**(E7.14)** Let \( k \) and \( \ell \) be integers with \( k \geq 5 \). Show that \( W := \text{Sym}(k) \wr \text{Sym}(\ell) \) has a unique minimal normal subgroup, and give its isomorphism type.