UNIVERSIDAD DE COSTA RICA | ESCUELA DE MATEMATICA

Tópicos en la teoría de grupos | Examen Parcial 1

I Semestre 2014 | Nick Gill

Instructions: You may use any of the results covered in the lecture notes, including in exercises. Make sure that you state clearly the results that you use.

If a question asks you to prove a result from lectures, then you should sketch it as fully as possibly, explicitly stating all other results that you use.

(1) Let K be a group. Show that we can define an action of the direct product $K \times K$ on the set K by

$$a^{(x,y)} := x^{-1}ay$$

for all $a \in K$ and $(x, y) \in K \times K$. Show that the action is transitive and find the stabilizer of the element 1. When is the action faithful?

Answer.
Claim: We have an action.

Proof. Observe, first, that $a^{(1,1)} = a$ for all $a \in K$. Observe, second, that

$$\begin{aligned}(a^{(x_1,y_1)})^{(x_2,y_2)} &= (x_1^{-1}ay_1)^{(x_2,y_2)} = x_2^{-1}(x_1^{-1}ay_1)y_2 \\ &= (x_1x_2)^{-1}a(y_1y_2) = a^{(x_1x_2,y_1y_2)} = a^{(x_1,y_1)(x_2,y_2)}\end{aligned}$$

□

Claim: The action is transitive.

Proof. Let $a, b \in K$. Then $a^{(1,a^{-1}b)} = b$ and we are done. □

The stabilizer of the element 1 is the group

$$\begin{aligned}H &:= \{(a, b) \in K \times K \mid a^{-1} \cdot 1 \cdot b = 1\} \\ &= \{(a, a) \in K \times K\}.\end{aligned}$$

Claim: The action is faithful if and only if $Z(K) = \{1\}$

Proof. Write L for the kernel of the action and observe that $L \leq H$, the stabilizer of 1, described above. Then

$$\begin{aligned}L &:= \{(a, a) \mid a^{-1}xa = x \text{ for all } x \in K\} \\ &= \{(a, a) \mid a \in Z(K)\}.\end{aligned}$$

The claim follows. □

(2) Describe the conjugacy classes of Alt(6). In particular, calculate the total number of conjugacy classes, list a representative of each, and calculate the size of each.

Answer. The conjugacy classes of Sym(6) are indexed by the partitions

$$1^6, 1^42^1, 1^22^2, 2^3, 1^33^1, 1^12^13^1, 3^2, 1^24^1, 2^14^1, 1^15^1, 6^1.$$

Of these, the following partitions correspond to conjugacy classes lying inside Alt(6):

$$1^6, 1^22^2, 1^33^1, 3^2, 2^14^1, 1^15^1.$$

We must ascertain which of these split into 2 inside Alt(6). Recall that a conjugacy class C containing an element g splits in 2 if and only if $C_{\mathrm{Sym}(6)}(g) \leq \mathrm{Alt}(6)$. Consider each class in turn:

(1^6) Clearly this cannot split!

($1^2 2^2$) Consider $g = (1,2)(3,4)$. This is centralized by $h = (1,2) \notin \mathrm{Alt}(6)$, so the class does not split.

($1^3 3^1$) Consider $g = (1,2,3)$. This is centralized by $h = (4,5) \notin \mathrm{Alt}(6)$, so the class does not split.

(3^2) Consider $g = (1,2,3)(4,5,6)$. This is centralized by $h = (1,4)(2,5)(3,6) \notin \mathrm{Alt}(6)$, so the class does not split.

($2^1 4^1$) Consider $g = (1,2)(3,4,5,6)$. This is centralized by $h = (1,2) \notin \mathrm{Alt}(6)$, so the class does not split.

($1^1 5^1$) Consider $g = (1,2,3,4,5)$. The centralizer of g is $\langle g \rangle < \mathrm{Alt}(6)$, thus this conjugacy class splits into two.

Let us summarize the results:

Number	Cycle type	Size of class	Representative
1	1^6	1	(1)
2	$1^2 2^2$	45	$(1,2)(3,4)$
3	$1^3 3^1$	40	$(1,2,3)$
4	3^2	40	$(1,2,3)(4,5,6)$
5	$2^1 4^1$	90	$(1,2)(3,4,5,6)$
6	$1^1 5^1$	72	$(1,2,3,4,5)$
7	$1^1 5^1$	72	**

To complete the answer we must find a representative of the final conjugacy class. It must be an element of type $1^1 5^1$ that is not conjugate to $g = (1,2,3,4,5)$ in Alt(6). let h be of type $1^1 5^1$. The set of elements in Sym(6) that conjugate g to h is a coset of $C_G(g)$, thus they either all lie in Alt(6), or they all lie in $\mathrm{Sym}(6) \setminus \mathrm{Alt}(6)$. We can, therefore, take our representative to be $h = g^t$ where $t = (1,2)$ and we obtain,

$$h = (1,2)(1,2,3,4,5)(1,2) = (1,3,4,5,2).$$

(3) Let k, n be integers with $1 \leq k \leq \frac{n}{2}$ and let $G = \mathrm{Sym}(n)$. Let H be the setwise stabilizer in G of a set of size k in $\{1, \ldots, n\}$. Recall that $H \cong \mathrm{Sym}(k) \times \mathrm{Sym}(n-k)$. Let $K := H \cap \mathrm{Alt}(n)$.
   (a) Prove that, if $n \geq 3$, then $|H : K| = 2$.
   (b) Prove that, if $n \geq 3$ and $k = 1$, then $K = \mathrm{Alt}(n-1)$.
   (c) Prove that, if $n \geq 3$ and $k > 1$, then $K \cong (\mathrm{Alt}(k) \times \mathrm{Alt}(n-k)) \rtimes C_2$.
   (d) Assume that $n \geq 8$ and describe the socle of K.

Describe $H \cap \mathrm{Alt}(n)$. Describe the socle of $H \cap \mathrm{Alt}(n)$.

Answer.

(a) **Claim 1:** H contains an odd element g.

Proof. Since $n \geq 3$, $n - k \geq 2$. Thus $\mathrm{Sym}(n-k)$ contains a transposition g_2. Thus we can take $g = (1, g_2) \in \mathrm{Sym}(k) \times \mathrm{Sym}(n-k)$. □

Claim 2: $|H : K| = 2$.

Proof. Since $|\mathrm{Sym}(n) : \mathrm{Alt}(n)| = 2$, we know that $|H : K| \leq 2$. If $|H : K| \neq 2$, then $H = K$, but this contradicts Claim 1 and we are done. □

(b) Suppose that $k = 1$. Then $H \cong \mathrm{Sym}(n-1)$ and, since $K \geq \mathrm{Alt}(n-1)$, Claim 2 implies that $H = \mathrm{Alt}(n-1)$.

(c) Suppose that $k \geq 2$. It is clear that $K > K_0 := \mathrm{Alt}(k) \times \mathrm{Alt}(n-k)$. Thus Claim 2 implies that $|K : K_0| = 2$. Now let $g = (g_1, g_2) \in \mathrm{Sym}(k) \times \mathrm{Sym}(n-k)$ where g_1 and g_2 are both transpositions. Then $g \notin K_0$, but $g \in K$, since g is the product of two transpositions. Thus $K = \langle K_0, g \rangle$. Furthermore, since $|K : K_0| = 2$, K_0 is a normal subgroup of K and we conclude that $K = K_0 \rtimes \langle g \rangle$.

(d)

($k = 1$) Then $K = \mathrm{Alt}(n-1)$, a simple group, and $K = \mathrm{soc}(K)$.

($k = 2$) Then $\mathrm{Alt}(k)$ is trivial and so $K_0 \cong \mathrm{Alt}(n-2)$ and $K \cong \mathrm{Sym}(n-2)$, an almost simple group. Then $\mathrm{soc}(K) = K_0 = \mathrm{Alt}(n-2)$.

($k = 3$) Then $\mathrm{Alt}(k) \cong C_3$, a simple group and so $K_0 \cong \mathrm{Alt}(3) \times \mathrm{Alt}(n-3)$, is a direct product of two simple groups, and so must be contained in the socle. Since K is not a direct product of K_0 with C_2, we conclude that $\mathrm{soc}(K) = K_0$.

($k = 4$) Then $\mathrm{Alt}(k)$ is not simple. If $n \geq 8$, then $\mathrm{Alt}(n-k)$ is simple and so $\mathrm{soc}(K) = K_4 \times \mathrm{Alt}(n-k)$. If $n = 8$, then $\mathrm{soc}(K) = K_4 \times K_4$.

($k \geq 5$) Then $\mathrm{Alt}(k)$ and $\mathrm{Alt}(n-k)$ are both simple and so $K_0 \cong \mathrm{Alt}(k) \times \mathrm{Alt}(n-3)$, is a direct product of two simple groups, and so must be contained in the socle. Since K is not a direct product of K_0 with C_2, we conclude that $\mathrm{soc}(K) = K_0$.

(4) Do **ONE** of the following:

(a) Describe how to construct an exceptional automorphism of $\mathrm{Alt}(6)$ (i.e. an automorphism that is not induced by conjugation by an element of $\mathrm{Sym}(6)$); sketch a proof that the automorphism you have constructed is indeed exceptional;

(b) Let H and K be groups and suppose that H acts on a set Δ and K acts on a set Γ.

- Describe $K \wr_\Delta H$;
- Describe the product action of $K \wr_\Delta H$ on Γ^Δ;
- Prove that if K acts primitively but not regularly on Γ, if Δ is finite, and if H acts transitively on Δ, then the product action is primitive.

Answer. This is book work, so an answer will not be provided.

(5) (a) Let $\Omega = \{1, \ldots, 6\}$ and let G be the unique subgroup of $\mathrm{Sym}(6)$ such that

- G is isomorphic to $\mathrm{Sym}(2) \wr \mathrm{Sym}(3)$; and
- there is a G-congruence $\sim$ with associated blocks

$$B_1 = \{1, 2\}, B_2 = \{3, 4\} \text{ and } B_3 = \{5, 6\}.$$

Prove that G is maximal in $\mathrm{Sym}(6)$.

(b) More generally, suppose that H is a subgroup of $\mathrm{Sym}(n)$ such that

- H is isomorphic to $\mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$ for some integers $k, l \geq 2$; and
- there is a H-congruence $\sim$ with ℓ associated blocks each of size k.

Prove that H is maximal in $\mathrm{Sym}(n)$.

Answer. (a) This could be a corollary of (b), but here is a direct proof. Suppose that $G > M > \mathrm{Sym}(6)$. The group G has index 15 in $\mathrm{Sym}(6)$, thus M must have index 5 or 3. Now the action of $\mathrm{Sym}(6)$ on the cosets of M is transitive and has an associated homomorphism $\phi : \mathrm{Sym}(6) \to \mathrm{Sym}(k)$ where $3 \leq k \leq 5$. This action cannot be faithful (by considering orders), and the only non-trivial subgroups of $\mathrm{Sym}(6)$ are $\mathrm{Alt}(6)$ and $\mathrm{Sym}(6)$. But if either of these were the kernel of ϕ, then the image of ϕ would have order at most 2, in particular this image could not be a transitive subgroup of $\mathrm{Sym}(k)$. We are done.

(b) The group H is clearly transitive, so cannot lie inside an intransitive group $\mathrm{Sym}(k) \times \mathrm{Sym}(n-k)$. On the other hand H contains a transposition, so (by a result in exercises) the only primitive subgroup that contains H is $\mathrm{Sym}(n)$.

Thus, if M is a subgroup such that $H < M < \mathrm{Sym}(n)$, then M is imprimitive. Suppose that $\sim'$ is a non-trivial M-congruence and let B' be an associated block.

Claim: B' is a union of blocks associated with $\sim$.

Proof. Suppose not. Then there is a pair $g, h \in \Omega$ such that $g \sim h$, $g \in B'$ and $h \notin B'$. Now G contains the tranposition (g, h) and so must move the block B'. But this implies that $|B'| = 1$ which is a contradiction. □

Claim: B' is a block for $\sim$.

Proof. Suppose not. Then there are two distinct $\sim$-blocks B_1 and B_2 inside B'. Let B_3 be a $\sim$-block that is not in B'. Now G contains an element that fixes all elements of B_1 and sends all elements of B_1 to B_2. This is a contradiction. □

Since B' was arbitrary, we conclude that all blocks for $\sim'$ are blocks for $\sim$. But this means that $\sim = \sim'$. Then (by lectures) M is a subgruop of a group isomorphic to G which is a contradiction.

(6) Let $\Omega = \{1, \ldots, 6\}$ and let G be the unique subgroup of $\mathrm{Sym}(6)$ such that
- G is isomorphic to $\mathrm{Sym}(2) \wr \mathrm{Sym}(3)$; and
- there is a G-congruence $\sim$ with associated blocks

$$B_1 = \{1,2\}, B_2 = \{3,4\} \text{ and } B_3 = \{5,6\}.$$

(a) Write down a set of permutations that generate G.

(b) Let $Z(G)$ be the centre of G; show that $|Z(G)| = 2$ and write down the unique $g \in Z(G)\backslash\{1\}$.

A *partition* of Ω is a set of disjoint subsets of Ω whose union is equal to Ω. Observe that $\lambda := \{B_1, B_2, B_3\}$ is a partition of Ω. Let μ be another partition of Ω; we say that μ is *orthogonal* to λ if μ contains two sets C_1, C_2 each of size 3 and, for all $1 \leq i \leq 3$ and $1 \leq j \leq 2$, $|B_i \cap C_j| = 1$.

(c) Write down the four partitions of Ω that are orthogonal to λ. Call the set of these four partitions $\lambda^\perp$.

(d) Show that G acts on $\lambda^\perp$ via

$$\Big\{\{C_{11}, C_{12}, C_{13}\}, \{C_{21}, C_{22}, C_{23}\}\Big\}^g = \Big\{\{C_{11}^g, C_{12}^g, C_{13}^g\}, \{C_{21}^g, C_{22}^g, C_{23}^g\}\Big\}$$

where $g \in G$ and $C_{11}, C_{12}, C_{13}, C_{21}, C_{22}, C_{23} \in \{1, \ldots, 6\}$.

(e) Let $\phi : G \to \mathrm{Sym}(4)$ be the homomorphism associated with the action of G on $\lambda^\perp$. Show that
- $\phi((1,3,5)(2,4,6))$ is a 3-cycle;
- $\phi((1,2)(3,5,4,6))$ is a 4-cycle.

(f) Prove that if g is any 3-cycle in $\mathrm{Sym}(4)$ and h is any 4-cycle in $\mathrm{Sym}(4)$, then $\langle g, h\rangle = \mathrm{Sym}(4)$. Conclude that $G/Z(G) \cong \mathrm{Sym}(4)$.

(g) Describe $G \cap \mathrm{Alt}(6)$ and prove that $G \cong Z(G) \times \mathrm{Sym}(4)$.

Answer. It is convenient to set some notation. We let

$$B = \langle (1,2), (3,4), (5,6) \rangle,$$
$$h = (1,3,5)(2,4,6), H = \rangle h, (1,3)(2,4)\rangle.$$

Recall that B is a normal subgroup of G that is isomorphic to $\mathrm{Sym}(2) \times \mathrm{Sym}(2) \times \mathrm{Sym}(2)$. Recall that $H \cong \mathrm{Sym}(3)$ and that $G = B \rtimes H$.

(a) There are many possibilities for this. For instance

$$G = \langle (1,2), (3,4), (5,6), (1,3)(2,4), (1,5)(2,6) \rangle.$$

(In fact you could miss out $(3,4)$ and $(5,6)$ if you wanted.)

(b) The element $g = (1,2)(3,4)(5,6)$ is central. We need to show that $Z(G) = \{1, g\}$. Recall that every element of G can be written uniquely as bh for some $b \in B$ and $h \in H$. Consider an element $bh \in Z(G)$.

Since $H \cong \mathrm{Sym}(3)$ has trivial centre, we conclude that if $bh \in Z(G)$, then $h = 1$. Now let $b = (b_1, b_2, b_3) \in \mathrm{Sym}(2) \times \mathrm{Sym}(2) \times \mathrm{Sym}(2)$. If the b_i are not all identical, then $bh \neq hb$. The result follows.

(c)

$$A := \Big\{\{1,3,5\},\{2,4,6\}\Big\}, B := \Big\{\{1,3,6\},\{2,4,5\}\Big\}$$
$$C := \Big\{\{1,4,5\},\{2,3,6\}\Big\}, D := \Big\{\{1,4,6\},\{2,3,5\}\Big\}$$

(d) We need to check that if $g \in G$ and $\mu \in \lambda^\perp$, then $\mu^g \in \lambda^\perp$. Once this is done, the two axioms are a formality. So, let $\mu = \{C_1, C_2\}$. It is obvious that μ^g is a set consisting of two subsets, each of size 3 that partition $\{1, \ldots, 6\}$. Now observe that

$$|C_i^g \cap B_j| = |C_i \cap B_j^{g^{-1}}| = |C_i \cap B_k| = 1,$$

as required. (Here we write B_k for $B_j^{g^{-1}}$ and we use the fact that G preserves the set of blocks of $\sim$.)

(e) Using the notation of (c) we confirm that

$$\phi((1,3,5)(2,4,6)) = (B, C, D)$$
$$\phi((1,2)(3,5,4,6)) = (A, B, D, C).$$

(f) Let $H = \langle g, h \rangle$. Since g has order 3 and h has order 4 we know that H has order 12 or 24. We assume that $|H| = 12$ and prove a contradiction. Observe that $C_{\mathrm{Sym}(4)}(g) = \langle g \rangle$ In particular g, g^h, g^{h^2}, g^{h^3} are all distinct. One can check, in addition, that these are all distinct from g^{-1}. We conclude that H contains all eight 3-cycles that are contained in $\mathrm{Sym}(4)$. On the other hand $N_{\mathrm{Sym}(4)}(\langle h \rangle)$ is group of order 8. Thus, in particular h^g is an order 4 element outside $\langle h \rangle$. Counting elements we find that we have at least 13 elements in H and we are done.

This result implies that the homomorphism $\phi : G \to \mathrm{Sym}(4)$ is surjective (since it contains a 3-cycle and a 4-cycle in its image). Thus we must show that $\ker(\phi) = Z(G)$. Since, by order considerations, we know that $|\ker(\phi)| = 2$, it is enough to show that $g = (1,2)(3,4)(5,6)$ lies in the kernel of ϕ. One checks directly that

$$A^g = A, B^g = B, C^g = C \text{ and } D^g = D.$$

(g) Let $K = G \cap \mathrm{Alt}(6)$ and let g be the non-identity element in $Z(G)$. Since $g \in K$ and $g \notin \mathrm{Alt}(6)$, we conclude that K is a proper subgroup of G; indeed it is a subgroup of index 2 and order 24.

Observe next that the two elements listed in (c) both lie in K. Thus, by restricting the action of G on $\lambda^\perp$ to K we obtain an action whose associated homomorphism is onto $\mathrm{Sym}(4)$. Since $|\mathrm{Sym}(4)| = 24$ we conclude that ϕ is an isomorphism and $K \cong \mathrm{Sym}(24)$. Now $K \cap Z(G) = \{1\}$, both K and $Z(G)$ are normal subgroups of G, and so $G \geq K \times Z(G)$. Checking orders we find that $G = K \times Z(G) \cong \mathrm{Sym}(2) \times \mathrm{Sym}(4)$, as required.

(7) The last question concerns some properties of *p-groups*, i.e. finite groups G, such that $|G| = p^a$ for some prime p and positive integer a.

(a) Let G and H be finite p-groups, for some prime p. Suppose that G acts on H as an object from **Group**. Define

$$\mathrm{Fix}(G) := \{h \in H \mid h^g = h \text{ for all } g \in G\}.$$

Prove that $\mathrm{Fix}(G)$ is a non-trivial subgroup of H.

(b) Let G be a group with a finite normal subgroup K and let P be a Sylow p-subgroup of K. Show that $G = KN_G(P)$.

(c) Let G be a transitive subgroup of $\mathrm{Sym}(p^k m)$ where p is a prime, and k and m are positive integers. Show that if P is a Sylow p-subgroup of G, then each orbit of P has size at least p^k.

Answer. (a) Suppose that $h_1, h_2 \in \mathrm{Fix}(G)$ and let $g \in G$. Observe that

$$1 = 1^g = (h_1 \cdot h_1^{-1})g = h_1^g \cdot (h_1^{-1})^g = h_1 \cdot (h_1^{-1})^g.$$

Thus we conclude that $(h_1^{-1})^g = h_1^{-1}$ and so $h_1^{-1} \in \mathrm{Fix}(G)$. Similarly

$$(h_1 \cdot h_2)^g = h_1^g \cdot h_2^g = h_1 \cdot h_2$$

and so $h_1 \cdot h_2 \in \mathrm{Fix}(G)$. We conclude that $\mathrm{Fix}(G)$ is a subgroup of G.

Now consider the set of orbits associated with the action of G on H. All of these orbits have order divisible by p, except those in $\mathrm{Fix}(G)$. Since the orbits partition H, if $\mathrm{Fix}(G)$ were trivial, this would imply that $|H| \equiv 1 \pmod{p}$, a contradiction.

(b) G acts by conjugation on Ω, the set of Sylow p-subgroups of G. This action is transitive; indeed if we restrict this action and consider only the action of K on Ω, then it is already transitive.

If $P \in \Omega$, then the stabilizer of P in G is $N_G(P)$. Furthermore, for every $g \in G$, the coset $N_G(P)g$ consists of the set of elements h in G such that $P^h = P^g$. Since K is transitive, we conclude that K contains an element in every coset $N_G(P)g$ and so, in particular $N_G(P)K = G$.

(c) If a Sylow p-subgroup P has an orbit of size less than p^k, then there is an element $\omega \in \Omega$ such that the stabilizer P_ω has order greater than $|P|/p^k$. Since $P_\omega \leq G_\omega$, this implies, in particular that $|G|/|G_\omega$ is not divisible by p^k. But, since G is transitive, this contradicts the Orbit-Stabilizer theorem.