14. Isometries and Witt's Lemma

(E14.1) Let β be a σ-Hermitian, or alternating form, with radical $\mathrm{Rad}(V)$. Prove that the natural map $V \to V/\mathrm{Rad}(V)$ is an isometry. What happens if we ask the same question with β replaced by a quadratic form Q?

(E14.2*)Let U_1 and U_2 be subspaces of a vector space V having the same dimension. Show that there is a subspace W of V which is a complement for both U_1 and U_2.

(E14.3) $\dim(X^{\perp}) \geq n - \dim(X)$ with equality if and only if $\beta\,|_X$ is non-degenerate.

(E14.4) Suppose that (V, Q) is a hyperbolic line containing two elements x, y such that (x, y) is a hyperbolic pair and $Q(x) = 0$. Then there exists an element z such that (x, z) is a hyperbolic pair and $Q(x) = Q(z) = 0$.

(E14.5) Check that $h \oplus h'$ is an isometry.

(E14.6*) Let (V, κ) be a formed space. Then the Witt index and the isomorphism class of a maximal anisotropic subspace are determined.

(E14.7*) Let (V, κ) be a formed space. Any maximal totally isotropic/ totally singular subspaces in V have the same dimension. This dimension is equal to the Witt index.

(E14.8) The norm and trace functions are surjective.

(E14.9*) Let $a, b \in k^*$. For all $c \in k$, there exist $x, y \in k$ with $ax^2 + by^2 = c$.

(E14.10*)Prove the result for $\dim(V) = 2 = \mathrm{char}(k)$.

(E14.11) Prove the final assertion.