13. Forms and polar spaces

In this section V is a vector space over a field k.

13.1. Sesquilinear forms. A *sesquilinear* form on V is a function

$$\beta : V \times V \to k$$

for which there exists $\sigma \in \mathrm{Aut}(k)$ such that

(1) $\beta(c_1x_1 + c_2x_2, y) = c_1\beta(x_1, y) + c_2\beta(x_2, y)$ for all $c_1, c_2 \in k$ and $x_1, x_2, y \in V$;
(2) $\beta(x, c_1y_1 + c_2y_2) = c_1^\sigma\beta(x, y_1) + c_2^\sigma\beta(x, y_2)$ for all c_1, c_2 in k and $x, y_1, y_2 \in V$.

In this case we say that β is *σ-sesquilinear*. If $\sigma = 1$, then k is a field and β is *bilinear*. We define

(1) The *left radical* of β is $\{x \in V \mid \beta(x, y) = 0, \forall y \in V\}$.
(2) The *right radical* of β is $\{y \in V \mid \beta(x, y) = 0, \forall x \in V\}$.

(E13.1*) *Prove that the left and right radicals are subspaces.*

(E13.2*) *Prove that if $\dim V < \infty$, then the left and right radicals have the same dimension. Give a counter-example to this assertion when $\dim V = \infty$.*

From here on we will assume that $n := \dim V < \infty$. We call β *non-degenerate* if its left and right radicals are trivial.

Recall that a *duality* of $\mathrm{PG}_{n-1}(k)$ is a weak automorphism that maps a subspace of dimension d to a subspace of dimension $n-d$. We can construct a duality from a non-degenerate sesquilinear form β as follows: for $y \in V$ define

$$\beta_y : V \to k, \ x \mapsto \beta(x, y).$$

Observe that the map $V \to V^*, y \mapsto \beta_y$ is a σ-semilinear bijection, and so induces an isomorphism $\mathrm{PG}(V) \to \mathrm{PG}(V^*)$. Now composing with the inverse of the 'annihilator map', $U \to U^\dagger$, which we have seen already, we obtain the duality

$$\mathrm{PG}(V) \to \mathrm{PG}(V), U \mapsto U^\perp := \{x \in V \mid \beta(x, y) = 0 \text{ for all } y \in U\}. \tag{16}$$

(E13.3*) *Check that this is a duality*

Theorem 13.1. *If $n \geq 3$, then any duality Δ of $\mathrm{PG}(V)$ has form $U \to U^\perp$ where $U^\perp$ is defined via (16) for some non-degenerate sesquilinear form β.*

Proof. Proposition 11.3 implies that $\Delta = st^{-1}$ where s is induced by a semilinear bijection $\phi : V \to V^*$ and $t : U \to U^\dagger$ is the annihilator map. Now set

$$\beta : V \times V \to k, (x, y) \mapsto x^{y\phi}$$

and the result follows. □

Let us fix β to be a σ-sesquilinear form on V, and let $U \mapsto U^\perp$ be the associated duality, given at (16). We say that β is *reflexive* if $\beta(x, y) = 0$ implies $\beta(y, x) = 0$. For a reflexive form, the left and right radicals coincide and we shall just call this subspace the *radical* of β,

$$\mathrm{Rad}(\beta) := \{v \in V \mid \beta(v, w) = 0 \text{ for all } w \in V\}.$$

Clearly a reflexive form β is non-degenerate if and only if $\mathrm{Rad}(\beta) = \{0\}$.

Observe that $U \to (U^\perp)^\perp$ is a collineation of $\mathrm{PG}(V)$. A *polarity* is a duality with $U = (U^\perp)^\perp$ for all $U \leq V$.

Lemma 13.2. *Let β be non-degenerate. The duality (16) is a polarity if and only if β is reflexive.*

Proof. The form β is reflexive if and only if

$$x \in \langle y\rangle^\perp \Longrightarrow y \in \langle x\rangle^\perp.$$

Thus if β is reflexive, then $U \leq U^{\perp\perp}$ for all $U \leq V$. Now, since β is non-degenerate,

$$\dim(U^{\perp\perp}) = \dim(V) - \dim(U^\perp) = \dim(U),$$

and so $U = U^{\perp\perp}$ for all U.

For the converse, given a polarity $\perp$, if $y \in \langle x\rangle^\perp$, then $x \in \langle x\rangle^{\perp\perp} \leq \langle y\rangle^\perp$ and we are done. □

We say that β is

(1) *σ-Hermitian*, where $\sigma \in \mathrm{Aut}(k)$, if $\beta(y,x) = \beta(x,y)^\sigma$ for all $x, y \in V$;
(2) *symmetric*, if $\beta(y,x) = \beta(x,y)$ for all $x, y \in V$;
(3) *alternating*, if $\beta(x,x) = 0$ for all $x \in V$;
(4) *skew-symmetric*, if $\beta(x,y) = -\beta(y,x)$ for all $x, y \in V$.

Note: if we say 'β is σ-Hermitian', we will implicitly assume that $\sigma \neq 1$, otherwise we would say that 'β is symmetric'. We record a number of easy observations in the next lemma.

Lemma 13.3. (1) *If β is σ-Hermitian, then $\sigma^2 = 1$ and $\beta(x,x) \in \mathrm{Fix}(\sigma)$ for all $x \in V$;*
(2) *If β is alternating, symmetric or skew-symmetric, then β is bilinear;*
(3) *If $\mathrm{char}(k) = 2$ and β is alternating, then β is symmetric;*
(4) *If $\mathrm{char}(k) \neq 2$, then β is alternating if and only if β is skew-symmetric.*
(5) *If β is σ-Hermitian, symmetric, alternating or skew-symmetric, then β is reflexive.*

Proof. (1) is easy. For (3) and (4) assume that β is alternating and observe that, for $x, y \in V$,

$$0 = \beta(x+y, x+y) = \beta(x,x) + \beta(x,y) + \beta(y,x) + \beta(y,y) = \beta(x,y) + \beta(y,x).$$

and the statements follows. For (2) and (5) the result is obvious unless β is alternating. But in that case, (3) and (4) imply that β is either symmetric or skew-symmetric, and the result follows. □

Theorem 13.5, proved below, is a partial converse to (5).

13.2. **Matrices and the classification of forms.** Let us fix a basis $\mathcal{B}$ for V and let β be a σ-sesquilinear form. It is easy to see that, there exists a matrix A such that, with respect to $\mathcal{B}$,

$$\beta(x,y) = x^T \cdot A \cdot y^\sigma.$$

We call A *the matrix for β with respect to $\mathcal{B}$*.[38]

The following proposition connects properties of β to properties of A.

Proposition 13.4. *Let β be a σ-sesquilinear form and A the matrix for β with respect to some basis.*
(1) *β is non-degenerate $\iff \mathrm{rank}(A) = n$;*
(2) *β is σ-Hermitian $\iff \sigma^2 = 1 \neq \sigma$ and $A = (A^T)^\sigma$;*
(3) *β is symmetric $\iff \sigma = 1$ and $A = A^T$;*
(4) *β is alternating $\iff \sigma = 1$, $A = -A^T$ and $A_{ii} = 0$ for $i = 1, \dots, n$;*
(5) *β is skew-symmetric $\iff \sigma = 1$ and $A = -A^T$;*

Proof. **(E13.4)** *Prove this.*

□

We are now ready to classify reflexive σ-sesquilinear forms. In the course of the proof we will encounter a matrix characterization of such a form.

Theorem 13.5. *Let $\beta : V \times V \to k$ be a reflexive σ-sesquilinear form. If $\dim(V/\mathrm{Rad}(\beta)) \geq 3$, then β is of one of the following types:*
(1) *alternating;*
(2) *symmetric;*
(3) *a scalar multiple of a σ-Hermitian form with $\sigma^2 = 1 \neq \sigma$.*

Proof. **1. Claim**: It is sufficient to prove the theorem for the case when β is non-degenerate.
Proof of claim: Suppose that $\beta : V \times V \to k$ is degenerate. Write R for the radical $\mathrm{Rad}(\beta)$. Then define the form

$$\beta_0 : V/R \times V/R \to k, (x+R, y+R) \mapsto \beta(x,y).$$

It is easy to check that β_0 is a well-defined, non-degenerate, reflexive σ-sesquilinear form. If we assume that the theorem is true for non-degenerate forms, then β_0 is one of the three listed types. Now, since $\beta(x,y) = \beta_0(x+R, y+R)$, β is also one of the three listed types and we are done.

Thus we assume from here on that β is non-degenerate.

2. Claim: $\{\lambda \in k \mid \lambda\lambda^\sigma = 1\} = \{\epsilon/\epsilon^\sigma \mid \epsilon \in k\}$.

[38]Note that, if $y = (y_{ij})$, a matrix with entries in the field k, then we define $y^\sigma := (y_{ij}^\sigma)$.

(E13.5*) *Prove the claim.*

3. Finish the proof. Let A be the matrix for β with respect to some fixed basis $\mathcal{B}$. For $x_1, \dots, x_l \in V$, define

$$[x_1, \dots, x_l] := \{y \in V \mid y^t x_1 = y^T x_2 = \cdots = y^T x_l = 0\}$$

Now define Δ_0 to be the polarity of $\mathrm{PG}(V)$ that, for $x_1, \dots, x_n \in V$, does

$$\langle x_1, \dots, x_n \rangle \longleftrightarrow [x_1, \dots, x_n].$$

(E13.6*) *Prove that this is a polarity.*

Next let Δ be the polarity associated with β. Thus, if $y \in V$, then

$$\begin{aligned} \langle y \rangle^\Delta &= \{x \in V \mid x^T A y^\sigma = 0\} \\ &= [Ay^\sigma]. \end{aligned} \tag{17}$$

Now observe that $\Delta\Delta_0$ and $\Delta_0\Delta$ are collineations of $\mathrm{PG}(V)$ and so are induced by semilinear transformations on V, $\overline{\Delta\Delta_0}$ and $\overline{\Delta_0\Delta}$ respectively. Now (17) implies that

$$y\overline{\Delta\Delta_0} = cAy^\sigma \tag{18}$$

for some constant c. On the other hand, suppose that $y^T z = 0$. Let $x = A^{-T} y^\sigma$ and observe that

$$x^T A z^\sigma = (y^\sigma)^T A^{-1} A z^\sigma = 0.$$

We conclude that $[y]^\Delta = \langle A^{-T} y^\sigma \rangle$ and so

$$y\overline{\Delta_0\Delta} = dA^{-T} y^\sigma \tag{19}$$

for some constant d.

Let us calculate the composition $\overline{\Delta\Delta_0} \cdot \overline{\Delta_0\Delta}$:

$$ey\overline{\Delta\Delta_0} \cdot \overline{\Delta_0\Delta} = ecAy^\sigma \overline{\Delta_0\Delta} = edc^\sigma A^{-T} A^\sigma y^{\sigma^2}.$$

Clearly $\overline{\Delta\Delta_0} \cdot \overline{\Delta_0\Delta}$ induces the collineation Δ^2 and, since Δ is a polarity, $\Delta^2 = 1$. This implies that $\overline{\Delta\Delta_0} \cdot \overline{\Delta_0\Delta}$ lies in the kernel of the action of $\Gamma\mathrm{L}_n(k)$ on $\mathrm{PG}_{n-1}(k)$. By (E11.8) we know that this kernel is equal to the set of invertible scalar matrices, thus we conclude that $\sigma^2 = 1$ and $A^{-T} A^\sigma = cI$ for some constant c. We therefore obtain that

$$A = c(A^T)^{\sigma^{-1}} \tag{20}$$

for some $c \in k$. Now (20) implies, immediately that $A^T = cA^{\sigma^{-1}}$ and substituting this identity in we obtain

$$A = cc^{\sigma^{-1}} A. \tag{21}$$

Suppose, first that $\sigma = 1$. Then $c^2 = 1$. If $c = 1$, then $A = A^T$ and β is symmetric; if $c = -1$, then $A = -A^{-T}$ and β is skew-symmetric, hence alternating by Lemma 13.3.

Suppose next that $\sigma \neq 1$. By the claim there exists $e \in k^*$ with $e/\sigma(e) = c$. Then the form $e\beta$ has matrix $B = eA$ which satisfies $B = (B^T)^\sigma$ and so $e\beta$ is Hermitian as required. □

For those of you who think that one should never prove anything in linear algebra by taking a basis, you can refer to [Camb] for a (rather long) matrix-free proof of this result.

13.3. **Trace-valued forms.** Let k be a field and $\sigma \in \mathrm{Aut}(k)$ with $o(\sigma) \in \{1, 2\}$. Define

$$\mathrm{Fix}(\sigma) := \{c \in k \mid \sigma(c) = c\}$$
$$\mathrm{Trace}(\sigma) := \{c + c\sigma \mid c \in k\}.$$

The following exercises list the key properties of these subsets.

(E13.7) $\mathrm{Fix}(\sigma)$ *and* $\mathrm{Trace}(\sigma)$ *are both subfields of* k.

(E13.8) $\mathrm{Fix}(\sigma) = \mathrm{Trace}(\sigma)$ *unless* $\mathrm{char}(k) = 2$ *and* $\sigma = 1$, *in which case* $\mathrm{Trace}(\sigma) = \{0\}$.

If β is a σ-sesquilinear form, then we call β *trace-valued* if $\beta(x, x) \in \mathrm{Trace}(\sigma)$ for all x. Recall that, by Lemma 13.3, $\beta(x, x) \in \mathrm{Fix}(\sigma)$. This, and (E13.8), immediately yield the following result.

Lemma 13.6. *A σ-sesquilinear form is not trace-valued if and only if* $\mathrm{char}(k) = 2$ *and β is symmetric and not alternating.*

In what follows we will study only trace-valued forms, and this will be enough for us to define and study all of the finite classical groups. One reason to avoid non-trace-valued forms is given by the following exercise. Recall that a field of characteristic 2 is called *perfect* if the map $x \mapsto x^2$ is an automorphism. In particular a finite field of characteristic 2 is perfect.

(E13.9) *Let* $\text{char}(k) = 2$ *and suppose that* k *is perfect. Let* β *be symmetric and define*

$$U := \{x \in V \mid \beta(x, x) = 0\}.$$

Then U *is a subspace of dimension at least* $n - 1$*.*

When we come to study isometries we shall see that this exercise implies that the isometry group of a non-trace-valued form cannot act *irreducibly* on the underlying vector space.

13.4. **Quadratic forms.** A *quadratic form* on V is a function $Q : V \to k$ such that

- $Q(cx) = c^2 Q(x)$ for all $c \in k, x \in V$;
- The function
$$\beta_Q : V \times V \to k, (x, y) \mapsto Q(x + y) - Q(x) - Q(y)$$
is a bilinear form.

The form β_Q is called the *polarization of* Q. Observe that β_Q is symmetric. If $\text{char}(k) = 2$, then it is also alternating (and so, in particular, β_Q is always trace-valued).

A quadratic form can be thought of as a homogeneous polynomial of degree 2 with coefficients in k. The next exercise makes this clear, as well as connecting quadratic forms to matrices.

(E13.10*) *Fix a basis* $\mathcal{B} = \{x_1, \dots, x_n\}$ *for* V *and let* $Q : V \to k$ *be a quadratic form. There is a matrix* A *such that* $Q(x) = x^T A x$*. Moreover*

$$A_{ij} = \begin{cases} \beta_Q(x_i, x_j), & \text{if } i < j, \\ Q(x_i), & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

The significance of quadratic forms depends on the characteristic of the field.

Suppose that $\text{char}(k)$ **is odd.** In this case the study of quadratic forms is equivalent to the study of symmetric bilinear forms. For, from every quadratic form Q, one obtains a symmetric bilinear form β_Q, and the next exercise shows that one can reverse this:

(E13.11) *If* $\text{char}(k) \neq 2$*, then* $Q(x) = \frac{1}{2}\beta_Q(x, x)$*.*

In particular a vector x satisfies $Q(x) = 0$ if and only if $\beta_Q(x, x) = 0$.

Suppose that $\text{char}(k) = 2$**.** Our restriction to the study of trace-valued forms means that, by studying alternating forms, we cover all symmetric forms in which we are interested. However we also choose to study quadratic forms because we obtain some interesting extra structure, as follows.

We know that, from every quadratic form Q, one obtains a symmetric, alternating bilinear form β_Q. However, in the reverse direction, suppose that β is a symmetric, alternating bilinear form with associated matrix B with respect to some basis β. Now define the matrix A via

$$A_{ij} = \begin{cases} B_{ij}, & \text{if } i < j, \\ 0, & \text{if } i > j. \end{cases}$$

We have not defined the diagonal on the matrix A - we can set it to be anything that we choose. Now define $Q(x) = x^T A x$.

(E13.12*) *Check that* Q *polarizes to* β*.*

Thus we find that many quadratic forms polarize to the same alternating form. In particular it is **not true** in general that a vector x satisfies $Q(x) = 0$ if and only if $\beta_Q(x, x) = 0$. We shall see that this fact results in the geometric behaviour of Q and β_Q being very different.

Let $Q : V \to k$ be a quadratic form. Recall the definition of the *radical* of β_Q,

$$\text{Rad}(\beta_Q) := \{v \in V \mid \beta_Q(v, w) = 0 \text{ for all } w \in V\}.$$

We define the *singular radical* of a quadratic form to be

$$\{v \in \text{Rad}(\beta_Q) \mid Q(v) = 0\}.$$

If the singular radical of Q is trivial, then we say that Q is *non-degenerate.*

(E13.13) *If* $\mathrm{char}(k) \neq 2$*, then* β_Q *is non-degenerate if and only if* Q *is non-degenerate.*

(E13.14*) *If* $\mathrm{char}(k) = 2$*,* k *is perfect, and* $Q : V \to k$ *is non-degenerate, then* $\dim(\mathrm{Rad}(\beta_Q)) \leq 1$*.*

13.5. **Formed spaces.** We write (V, β) (resp. (V, Q)) to mean a vector space equipped with a trace-valued non-degenerate reflexive σ-sesquilinear form β (resp. non-singular quadratic form Q). We call such a pair a *formed space.*

Then

- (V, β) is called *symplectic* if β is alternating;
- (V, β) is called *unitary* if β is σ-Hermitian;
- (V, β) is called *orthogonal* if β is symmetric and $\mathrm{char}(k) \neq 2$;
- (V, Q) is called *orthogonal*;

In fact we will not need to consider the third of these, since they are a subclass of the fourth. We will say a number of formed spaces are *of the same type* if they are all σ-Hermitian or all alternating or all symmetric.

Two formed spaces (V_1, Q_1) and (V_2, Q_2) are isomorphic if there exists an invertible linear map $A : V_1 \to V_2$ such that $Q_2 \circ A = Q_1$. A similar definition applies for forms β_1 and β_2. [39]

Let U be a vector subspace of a formed space (V, β), and write $\perp$ for the polarity defined by β. Then

- a vector $u \in V$ is *isotropic* if $\beta(u, u) = 0$;
- U is *totally isotropic* if $\beta(u, v) = 0$ for all $u, v \in U$ (equivalently, if $U \subseteq U^\perp$);
- U is *non-degenerate* if $\beta|_W$ is non-degenerate;
- U is a *hyperbolic line* if $U = \langle u, v\rangle$ and
$$\beta(u, u) = \beta(v, v) = 0, \ \beta(u, v) = 1.$$
The pair (u, v) is called a *hyperbolic pair.* (Notice that u and v must be linearly independent, so $\dim(U) = 2$.)

Let U be a vector subspace of a formed space (V, Q), and write $\perp$ for the polarity defined by the polarized form β_Q. Then the above definitions all apply with respect to the polarized form β_Q. In addition

- a vector $u \in V$ is *singular* if $Q(u) = 0$;
- U is *totally singular* if $Q(u) = 0$ for all $u \in U$.

We are working towards a classification of formed spaces in which we build them up from smaller spaces. We need to define what me mean by "building up." Let $(U_1, \beta_1), \dots, (U_\ell, \beta_\ell)$ be formed spaces of the same type. Define *the orthogonal direct sum* $U_1 \perp \cdots \perp U_\ell$ to be the vector space $V = U_1 \oplus \cdots \oplus U_\ell$ with associated form

$$\beta := \beta_1 \perp \cdots \perp \beta_\ell : (U_1 \perp \cdots \perp U_\ell) \times (U_1 \perp \cdots \perp U_\ell) \to k$$
$$((u_1, \dots, u_\ell), (v_1, \dots, v_\ell)) \mapsto \sum_{i=1}^{\ell} \beta(u_i, v_i).$$

Notice that, for each i, the space V has a subspace

$$V_i := 0 \perp \cdots 0 \perp U_i \perp 0 \cdots \perp 0$$

such that $\beta|_{V_i} = \beta_i$. We will often abuse notation and identify U_i and V_i, so that we can think of (V, β) as a direct sum of k of its subspaces.

An obvious analogous notion of orthgonal direct sum also exists for formed spaces involving a quadratic form.

(E13.15) *Any two hyperbolic lines of the same type are isomorphic (as formed spaces).*

(E13.16) *Suppose that* U, U' *(resp.* W, W'*) are isomorphic formed spaces of the same type. Then* $U \perp W$ *and* $U' \perp W'$ *are isomorphic formed spaces.*

Two more definitions:

- A formed space (V, β) is called *anisotropic* if $\beta(x, x) \neq 0$ for all $x \in V\backslash\{0\}$.

[39] If working with two symmetric space over a field of odd characteristic, one with a quadratic form, the other with a symmetric bilinear form, then there is an obvious notion of isomorphism which we will not write down here. Yet another reason to avoid studying symmetric bilinear forms in general.

- A formed space (V, Q) is called *anisotropic* if $Q(x) \neq 0$ for all $x \in V\backslash\{0\}$.

Theorem 13.7. *A formed space (V, β) (resp. (V, Q)) is the orthogonal direct sum of a number r of hyperbolic lines and an anisotropic space U.*

Proof. Define a function $f : V \to k$ which maps a vector x to $\beta(x, x)$ (resp. $Q(x)$). If V is anisotropic, then V does not contain a hyperbolic line, so $r = 0$ and U must equal V. Suppose then, that $f(v) = 0$ for some $v \in V\backslash\{0\}$. In the sesquilinear case, non-degeneracy implies that there exists $w \in V$ such that $\beta(v, w) \neq 0$. In the quadratic case, we claim there exists $w \in V$ such that $\beta(v, w) \neq 0$ where β is the polarized form. The claim follows because if no such w existed, then v would be in the radical of β and hence in the singular radical of κ which contradicts the fact that κ is non-singular.

We can replace w by a scalar multiple so that $\beta(v, w) = 1$. Observe that $\beta(v, w - \lambda v) = 1$ for all $\lambda \in k$. If we can find a value of λ for which $f(w - \lambda v) = 0$, then $\langle v, w\rangle$ will be a hyperbolic line. Consider three cases:

(1) If the form is alternating, then any value of λ works.
(2) If the form is σ-Hermitian, then
$$\begin{aligned} f(w - \lambda v) &= f(w) - \lambda\beta(v, w) - \lambda^\sigma\beta(w, v) + \lambda\lambda^\sigma f(v) \\ &= f(w) - (\lambda + \lambda^\sigma); \end{aligned}$$
and, since β is trace-valued, there exists $\lambda \in k$ with $\lambda + \lambda^\sigma = f(w)$ and we are done.
(3) If the form is quadratic, then
$$\begin{aligned} f(w - \lambda v) &= f(w) - \lambda\beta(w, v) + \lambda^2 f(v) \\ &= f(w) - \lambda \end{aligned}$$
and we choose $\lambda = f(w)$.

Now let W_1 be the hyperbolic line $\langle v, w - \lambda v\rangle$, and let $V_1 = W_1^\perp$.

(E13.17*) *$V = V_1 \oplus W_1$ and the restriction of the form to V_1 is non-degenerate (resp. non-singular).*

We conclude, by induction, that a decomposition of the given kind exists. □

In the next section we will prove Witt's Lemma, a corollary of which states that the number r and the isomorphism class of the space U, defined in Theorem 13.7, are invariants of the formed space (V, κ). We call r the *polar rank*, or the *Witt index*, of V, and U the *germ* of V.

It is worth taking a moment to reflect on the power of Theorem 13.7. Let us just consider the case where the form κ is σ-sesquilinear (there is a similar analysis when we have a quadratic form). Theorem 13.7 asserts that there is a basis for V such that
$$\beta(x, y) = x^t A y^\sigma$$
where the matrix A has form
$$\begin{pmatrix} A_{HL} & & & \\ & \ddots & & \\ & & A_{HL} & \\ & & & A_{An} \end{pmatrix}$$
where A_{HL} is a 2×2 matrix associated with a hyperbolic line, and A_{An} is a square matrix associated with an anisotropic form. Indeed we can be more precise:
$$A_{HL} = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \kappa \text{ is alternating;} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \text{otherwise.} \end{cases}$$

We shall spend some time in §14.2 studying the possibilities for A_{An}; in particular, we will see that it too has dimension at most 2.