15. Polar spaces

This section is something of a diversion, however it seems worthwhile to discuss polar spaces as they prefigure the work of Tits on *buildings* that enabled a uniform geometric understanding of the finite groups of Lie type.

15.1. **Abstract vs classical polar spaces.** Let (V, β) be a formed space. The polar space associated with V is the incidence structure $\mathcal{I} = (P_1, \dots, P_r, I)$ where P_i is the set of i-dimensional totally isotropic subspaces of V and two such subspaces are incident if and only if one is contained in the other.

Similarly if (V, Q) is a formed space, then the polar space associated with V is the incidence structure $\mathcal{I} = (P_1, \dots, P_r, I)$ where P_i is the set of i-dimensional totally singular subspaces of V and two such subspaces are incident if and only if one is contained in the other.

The objects in these polar spaces are referred to as *flats* and we use the terms points, lines, planes etc as for projective spaces.

Let $k = \mathbb{F}_q$. Let r be a positive integer. In the previous section we encountered a number of different formed spaces with polar rank r. We will use the following labels to refer to the polar space associated with each:

$$\mathrm{Sp}_{2r}, \mathrm{U}_{2r}, \mathrm{U}_{2r+1}, \mathrm{O}^+_{2\mathrm{r}}, \mathrm{O}_{2\mathrm{r}+1}, \mathrm{O}^-_{2\mathrm{r}+2}.$$

These are the *finite classical polar spaces of polar rank* r. Note that the subscript in each case gives the dimension of the formed space with which the polar space is associated.

Theorem 15.1. *Let* Γ *be a finite classical polar space of polar rank* r*. Then*

(Pol1) *Any flat, together with all the flats that it contains, is isomorphic to* $\mathrm{PG}_d(q)$ *for some* $d \leq r-1$.
(Pol2) *The intersection of any family of flats is a flat.*
(Pol3) *If* U *is a maximal flat and* p *is a point not in* U*, then the union of all lines joining* p *to points of* U *is a maximal flat* W *and* $U \cap W$ *is a hyperplane of both* U *and* W.
(Pol4) *There exist two disjoint maximal flats.*

(E15.1*) *Prove this.*

Any incidence structure satisfying (Pol1) to (Pol4) is a *thick abstract polar space.* The next theorem is due to Veldkamp and Tits and we will not prove it.

Theorem 15.2. *(Veldkamp-Tits) A thick abstract polar space with* $r \geq 3$ *is a classical polar space.*

An abstract polar space with $r = 2$ is called a *generalized quadrangle* - observe that an ordinary quadrilateral satisfies (Pol1) to (Pol4). There are many finite generalized quadrangles other than the classical ones.

The philosophy here is that a thick abstract polar space is some kind of analogue of thick abstract projective space, a concept we encountered in §11.5. The following definition and theorem is included to make this analogue more obvious. Let (P_1, P_2, I) be an incidence structure of points and lines and define the following three properties.

(BS1) Every line has at least 3 points
(BS2) No point is collinear with all the points of S.
(BS3) If x is a point that does not lie on a line L, then either
  (a) exactly one point of L is collinear with x, or
  (b) every point of L is collinear with x.

Theorem 15.3. *(Buekenhout-Shult) Any finite incidence structure satisfying (BS1) to (BS3) satsifies (Pol1) to (Pol4).*

15.2. **Spherical buildings.** Let us now discuss how these ideas can be pushed further.

Projective and polar spaces. In Section 11.5 we saw the connection between $\mathrm{PG}_n(q)$ and finite thick abstract projective spaces – the former are examples of the latter; indeed they are almost all possible examples of the latter.

In Section 15.1 we saw the connection between classical polar spaces and finite thick abstract polar spaces – the former are examples of the latter; indeed they are all possible examples of the latter except when $r = 2$.

Automorphisms. Recall next that the Fundamental Theorem of Projective Geometry states that, if $\dim(V) \geq 3$, then all collineations of $\mathrm{PG}(V)$ are induced by a semilinear transformation. Combining this with the Veblen-Young theorem we see that, in most cases, the automorphisms of a finite thick projective space are induced by a semilinar transformation of some associated vector space.

When one comes to finite thick abstract polar spaces the situation is similar. By the Tits-Veldkamp theorem above we can restrict our attention to the classical polar spaces where it is easy to see that semilinear transformations induce collineations:

(E15.2) *A semisimilarity of (V,κ) induces a collineation of the associated polar space $\mathcal{I}$. In particular* $\mathrm{SemiSim}(\kappa) \leq \mathrm{Aut}(\mathcal{I})$.

Now deep work of Tits [Tit74] implies that, in most cases, all collineations of a classical polar space are induced in this way – by a semilinear transformation of the associated vector space.

A generalization. In fact finite thick abstract projective spaces and finite thick abstract polar spaces can both be generalized to give the notion, introduced by Jacques Tits, of a *thick spherical building*. That is to say finite thick abstract projective spaces and finite thick abstract polar spaces can be thought of as examples of a finite thick spherical building.

Tits didn't just define these things; he also classified all finite thick spherical buildings, except 'when the rank is 2'.[42] In this pathological case one encounters the *generalized polygons* which include the generalized triangles (or projective planes) and generalized quadrangles, as well as generalized hexagons and octagons.

Automorphisms. The true significance of Tits' classification of the finite spherical buildings lies in their automorphism groups. We saw earlier that $\mathrm{Aut}(\mathrm{PG}_{n-1}(q)) = \mathrm{P}\Gamma\mathrm{L}_n(q)$ which, provided $n \geq 2$ or $q \geq 4$, is an almost simple group with simple normal subgroup $\mathrm{PSL}_n(q)$. In the next few sections we will see that the automorphism groups of the classical polar spaces are (generally speaking) almost simple groups with simple normal subgroup equal to a classical group. The beauty of Tits' classification is that the automorphism groups of the spherical buildings are (generally speaking) almost simple groups with simple normal subgroup equal to a finite group of Lie type. Thus the notion of a spherical building gives a uniform geometric description of the finite groups of Lie type.

15.3. **Connection to BN-pairs.** An alternative approach to buildings is through the idea of a BN-pair. Let us approach this subject by examining the (B,N)-structure of $\mathrm{GL}_n(k)$.[43] Let V be an n-dimensional vector space over a field k. Let $\{e_1,\dots,e_n\}$ be a basis for V and let $G=\mathrm{GL}_n(k)$.

(E15.3) *The chain of subspaces*

$$\langle e_1\rangle \subset \langle e_1,e_2\rangle \subset \langle e_1,e_2,e_3\rangle \subset \cdots \subset \langle e_1,\dots,e_{n-1}\rangle$$

is called a chamber.

Let B be the stabilizer in G of this chamber. What is B?

(E15.4) *Given a basis $\{e_1,\dots,e_n\}$, the corresponding* frame *is the set*

$$\mathcal{F} = \{\langle e_1\rangle, \langle e_2\rangle, \dots, \langle e_n\rangle\}.$$

Let N be the stabilizer in G of the given frame. What is N?

(E15.5) *Show that $G=\langle N,B\rangle$.*

Remark: In fact it is just as easy to show that $G=BNB$.

(E15.6) *Let $H=B\cap N$. Show that H is a normal subgroup of N.*

Remark: Note that H is the group of all diagonal matrices.

(E15.7) *The group $W := N/H$ is called the* Weyl group *of G. What well-known group is N/H isomorphic to?*

Remark: A *BN-pair* for a group G is a pair of subgroups B and N such that

(1) $G=\langle B,N\rangle$;
(2) $H=B\cap N \lhd N$;
(3) $W=N/H$ is generated by a set R of involutions such that, for $rH\in R$ and $n\in N$, then
  (3a) $rBnB \subset BnB \cup BrnB$;
  (3b) $rBr \neq B$.

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[42] We haven't defined what we mean by rank here, but we remark that for polar spaces it is precisely the same as the polar rank.

[43] This discussion class is taken from a course given by Michael Giudici. My thanks to him for letting me use it.

We call $|I|$ the *rank* of the BN-pair

(E15.8) *Let* $R := \{(1,2),(2,3),(3,4),\ldots,(n-1,n)\}$ *a generating set of size* $n-1$ *for the group* $\mathrm{Sym}(n)$. *Prove that, with this generating set, (3a) and (3b) are satisfied for* $GL_n(k)$*, i.e.* $GL_n(k)$ *has a* BN*-pair.*

Remark:

(1) It should be clear that, by taking the corresponding subgroups, we can see that $\mathrm{SL}_n(k)$, $\mathrm{PGL}_n(k)$ and $\mathrm{PSL}_n(k)$ also have BN-pairs.
(2) Tits has shown that given any group with a BN-pair, we can define a building on which G has a natural action. What is more, in this action, G is 'transitive on the pairs consisting of an apartment and a chamber contained in it' [Tit74, 3.2.6].
(3) Conversely Tits has shown that if a group G acts on a building so that it is 'transitive on the pairs consisting of an apartment and a chamber contained in it', then G has a BN-pair [Tit74, 3.11]. Thus the notion of a BN-pair and a building with this level of transitivity are closely linked.
(4) Finally Tits has shown that a finite building of 'irreducible type' and rank at least 3 is isomorphic to 'the building of a finite group of Lie type'. What is more such buildings admit transitive actions of the associated groups and we thereby have a full classification of those finite groups with a BN-pair of rank at least 3.
(5) Since the simple classical groups are 'groups of Lie type, they all have BN-pairs. Can you identify the groups B and N?

The remainder of the course will be spent studying the automorphisms of the finite classical polar spaces – the so called finite classical groups.