17. Unitary groups

(E17.1) Let β_1 and β_2 be non-degenerate σ-Hermitian forms defined on a n-dimensional vector space V over the field $k = \mathbb{F}_{q^2}$. Then $\mathrm{Isom}(\beta_1)$ and $\mathrm{Isom}(\beta_2)$ (resp. $\mathrm{Sim}(\beta_1)$ and $\mathrm{Sim}(\beta_2)$) are conjugate subgroups of $\mathrm{GL}_n(k)$. Furthermore $\mathrm{SemiSim}(\beta_1)$ and $\mathrm{SemiSim}(\beta_2)$ are conjugate subgroups of $\Gamma\mathrm{L}_n(k)$.

(E17.2*) $\mathrm{SU}_2(q) \cong \mathrm{SL}_2(q)$ and, moreover, the action of $\mathrm{SU}_2(q)$ on the set of points of the associated polar space is isomorphic to the action of $\mathrm{SL}_2(q)$ on the set of points of $\mathrm{PG}_1(q)$.

(E17.3*)Prove that if $\beta(x,y) = 0$, then there exists z with $\beta(x,z), \beta(y,z) \neq 0$.

(E17.4*)Prove that if $\beta(x,y) \neq 0$, then there exists z with $\beta(x,z) = \beta(y,z) = 0$.

(E17.5) Every unitary transvection is contained in a conjugate of the group Q defined in Lemma **??**.

(E17.6*)Complete this proof.

(E17.7*)Prove that $\mathrm{SU}_4(2)$ is generated by transvections.

(E17.8) Prove the result for $q = 2$ and $n \geq 4$.

(E17.9*)Prove that $\mathrm{PSU}_3(2) \cong E \rtimes Q$ where E is an elementary abelian group of order 9 and Q is a quaternion group of order 8.

18. Orthogonal groups

(E18.1) $O_1(q) = \{\pm I\}$ and $O_2^{\varepsilon}(q) \cong D_{2(q-\varepsilon 1)}$.

(E18.2*) Prove that

$$|\mathrm{SO}_n^{\varepsilon}(q)| = |\mathrm{PO}_n^{\varepsilon}(q)| = \frac{1}{(2, q-1)} |\mathrm{O}_n^{\varepsilon}(q)|.$$

(E18.3) Let q be odd. Show that $\mathrm{PSp}_{2m}(q)$ has $\lfloor \frac{m}{2} \rfloor + 1$ conjugacy classes of involutions, while $\mathrm{P}\Omega_{2m+1}(q)$ has m conjugacy classes of involutions.

(E18.4*)$SO_n^{\varepsilon}(q)$ contains a transvection if and only if q is even.

(E18.5*)Prove that this definition yields an index 2 subgroup when $\varepsilon = +$ by showing that, in the natural action of G on $\mathcal{U}_r$, the set of maximal totally singular subspaces, any reflection acts as an odd permutation on $\mathcal{U}_r$.

(E18.6) Calculate the order of $|\Omega_n^{\varepsilon}(q)|$ when $(n, q, \varepsilon) \neq (4, 2, +)$.