PRODUCTS OF ALTERNATING GROUPS

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1. Introduction

In this note we prove the following result.

Theorem 1. *Suppose that G is a finite group containing two proper subgroups H and K such that $G = H.K$. Suppose, moreover, that $H \cong \mathrm{Alt}(m)$ and $K \cong \mathrm{Alt}(n)$ with $m, n \geq 5$. Then one of the following holds:*

(1) *G is simple and $m, n \leq 72$;*
(2) *$G = \mathrm{Alt}(n+1)$;*
(3) *$G = \mathrm{Alt}(m) \times \mathrm{Alt}(n)$.*

Note that case (1) of Theorem 1 accounts for only a finite number of possible groups G. The theorem asserts, therefore, that given the suppositions on G (and barring a finite number of cases) G is isomorphic to the groups in cases (2) and (3).

Case (3) is, of course, entirely explicit and encompasses an infinite family of examples. Consider, on the other hand case (2): In this case K is the stabilizer of a point in the natural action of $G = \mathrm{Alt}(\ell)$ on ℓ points. Thus H must be a transitive subgroup of G and any factorization $G = K.H$ arises from the action of H on the set of cosets of one of its subgroups. The problem of classifying all such factorizations is equivalent to determining all (conjugacy classes of) subgroups of $G = \mathrm{Alt}(\ell)$.

Note that if one imposes the supposition that $H \cap K = \{1\}$ in the statement of Theorem 1, then one is asking what groups can be constructed as the *Zappa-Szép product* (or *knit product*) of two finite simple alternating groups. The extra supposition allows one to strengthen the statement of (2) to assert that $n + 1 = m!$ thereby making this case entirely explicit. To see why this extra assertion follows we use the remarks of the previous paragraph and observe that, given the extra supposition, the corresponding transitive action of H must be *simply* transitive.

1.1. Structure of this note.

In §2 we prove Proposition 2 which is a special case of Theorem 1 dealing with the situation where G is simple. To obtain an explicit, and reasonably low, upper bound in case (1) we made use of a result of Maróti which in turn depends on the Classification of Finite Simple groups (CFSG). An alternative would be to use results of Babai [Bab81] and [Pyb93] that do not depend on CFSG, but are not so sharp.

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The resulting CFSG-free version of Proposition 2 would start "Suppose that G is a *known* finite simple group that is sufficiently large..." and would not require the presence of case (1).

In §3 we prove Proposition 12 which is a general result concerning the structure of a group G that is equal to the product of two simple subgroups. Propositions 2 and 12 together yield a proof of Theorem 1. The proof of Proposition 12 depends on the Schreier Conjecture (that the outer automorphism group of any finite simple group is solvable), all known proofs of which depend on CFSG. Thus Theorem 1 uses CFSG in a crucial way.

1.2. **Acknowledgments.** This note was written in response to a question posted by Farrokh Shirjian on the website `MathOverflow`. I wish to thank Geoff Robinson for sharing his insight into this problem.

2. G is simple

In this section we prove a special case of Theorem 1 pertaining to the situation where G is simple. The primary result of this section is the following:

Proposition 2. *Suppose that G is a finite simple group containing two proper subgroups H and K such that $G = H.K$. Suppose, moreover that $H \cong \mathrm{Alt}(m)$ and $K \cong \mathrm{Alt}(n)$ with $m, n \geq 5$. Then one of the following holds:*

(1) $m, n \leq 72$;
(2) $G = \mathrm{Alt}(n+1)$.

We will prove Proposition 2 in a series of lemmas. Note that, for y an integer and p a prime we write $|y|_p$ to mean the largest power of p that divides y; we write $|y|_{p'}$ to mean $y/|y|_p$.

2.1. **G is a group of Lie type.** The first lemma is little more than an observation.

Lemma 3. *Let p be any prime and n any positive integer.*

$$|n!|_p \leq p^{\frac{n-1}{p}} \leq \begin{cases} 2^{n-1}, & p = 2 \\ 3^{\frac{n-1}{2}}, & p \neq 2 \end{cases}$$

Lemma 4. *Let p be a prime, n a positive integer. If $n \geq 73$, then $|\mathrm{Alt}(n)|_p^5 < |\mathrm{Alt}(n)|$.*

Proof. Lemma 3 implies that $|\mathrm{Alt}(n)|_p \leq 2^{n-1}$ and one can check that, if $n \geq 81$, then $2^{5n-5} < \frac{1}{2}n! = |\mathrm{Alt}(n)|$.

If $n \geq 41$, then $|\mathrm{Alt}(n)|_p^5 \leq 3^{\frac{5(n-1)}{2}} < |\mathrm{Alt}(n)|$ for every odd prime. Thus, to lower the bound below 81 we need only calculate the exact size of a Sylow 2-subgroup of $\mathrm{Alt}(n)$. A quick check confirms that, provided $n \geq 73$, the result holds. □

Lemma 5. *Let G be a finite simple group of Lie type. Then there is a prime p such that, if P is a Sylow p-subgroup of G, then $|P|^3 > |G|$.*

Proof. We can take p to be the charateristic of the field over which G is defined. The result then follows by checking the order formulae for the different groups of Lie type. □

Lemma 6. *Let G be a finite simple group of Lie type and let p be a prime such that, if P is a Sylow p-subgroup of G, then $|P|^3 > |G|$. Suppose that G contains two subgroups H and K such that $G = H.K$ and suppose that $|K|_p \geq |H|_p$. Then $|K|_p^5 \geq |K|$.*

Proof. Let $\ell_p = |G|_p/|K|_p$. Now observe that

$$\ell_p \cdot |K|_p = |G|_p \geq |G|^{\frac{1}{3}} \geq |K|^{\frac{1}{3}} \ell_p^{\frac{1}{3}}$$

and we obtain that $\ell_p^2 |K|_p^3 \geq |K|$. Now the result follows from observing that $\ell_p \leq |K|_p$. □

Putting these lemmas together we obtain the following:

Proposition 7. *Suppose that G is a finite simple group of Lie type containing two subgroups H and K such that $G = H.K$. Suppose, moreover that $H \cong \mathrm{Alt}(m)$ and $K \cong \mathrm{Alt}(n)$ with $5 \leq m \leq n$. Then $n \leq 72$.*

Proof. Observe that, since $m \leq n$, we have that $|H|_p \leq |K|_p$ for every prime. Now lemma 6 implies that $|K|_p^5 \geq |K|$ for some prime p. Then Lemma 4 gives the result. □

2.2. **Alternating groups.** To prove the main result of this section we need an upper bound on the size of a primitive group that does not contain the alternating group. There are a variety of possible bounds; we choose to use the following result of Maróti[Mar02].

Theorem 8. *If K is a primitive subgroup of $\mathrm{Sym}(\ell)$ that does not contain $\mathrm{Alt}(\ell)$, then $|K| < 3^\ell$.*

We will also need the following fact concerning multiply transitive actions of the alternative groups. A list of such actions can be found, for instance, in [Cam99].

Lemma 9. *If $\ell \geq 8$, then the only 2-transitive action of $\mathrm{Alt}(\ell)$ is the natural one on ℓ points.*

Proposition 10. *Suppose that $G = \mathrm{Alt}(\ell)$, and that H and K are two proper subgroups of G such that $G = H.K$. Suppose, moreover that $H \cong \mathrm{Alt}(m)$ and $K \cong \mathrm{Alt}(n)$ with $5 \leq m \leq n$. If $\ell \geq 23$, then $n = \ell - 1$.*

Proof. Let Ω_K be the largest orbit of K on the points $\Omega = \{1, \dots, \ell\}$, and let $k := |\Omega_K|$. The homomorphism associated to this action yields an embedding $\mathrm{Alt}(n) \to \mathrm{Alt}(k)$. We conclude that $n \leq k$.

If $k \leq \frac{\ell}{2}$, then we obtain that $|H| \leq |K| \leq \frac{1}{2}(\lfloor \frac{n}{2} \rfloor !)$. But in this case $|H| \cdot |K| < |G|$ which is a contradiction. Thus $k > \frac{\ell}{2}$.

By assumption the action of K on Ω_K is transitive. If the action is imprimitive, then K is isomorphic to a subgroup of a wreath product $\mathrm{Sym}(a) \wr \mathrm{Sym}(b)$ where $1 < a, b < k$ and $ab = k$. Since $a, b \leq \frac{k}{2}$ one immediately obtains that $n \leq \frac{k}{2}$ and, once again, $|H| \cdot |K| < |G|$, a contradiction.

Thus the action of K on Ω_K is primitive. Suppose that K does not contain $\mathrm{Alt}(\Omega_K)$. Then Theorem 8 implies that $|H| \leq |K| < 3^k \leq 3^\ell$. Thus $|H|.|K| \leq 9^\ell$ and, since $9^\ell < |\mathrm{Alt}(\ell)|$ for $\ell \geq 23$ we have a contradiction.

Thus $n = k$ and $K = \mathrm{Alt}(k)$. Since $k > \frac{\ell}{2}$, we conclude that K acts trivially on $\Omega \backslash \Omega_K$ and so K is precisely the pointwise stabilizer of $\Omega \backslash \Omega_K$ in G. This implies in particular that H acts $(\ell - n)$-transitively on Ω. Now Lemma 9 implies that, since H is a proper subgroup of G, $\ell - n = 1$. □

2.3. **Sporadic groups.**

We give the crudest possible result concerning sporadic groups.

Proposition 11. *Suppose that G is a sporadic simple group, and that H and K are two subgroups of G such that $G = H.K$. Suppose, moreover that $H \cong \mathrm{Alt}(m)$ and $K \cong \mathrm{Alt}(n)$ with $5 \leq m \leq n$. Then $n \leq 32$.*

Proof. We study the orders of the sporadic groups and note that none of these orders are divisible by 11^3. The result follows. □

3. The general case

In this section we prove the following result:

Proposition 12. *Suppose that G is a finite group containing two proper subgroups H and K such that $G = H.K$. Suppose, moreover that H and K are non-abelian simple. Then either G is simple or $G \cong H \times K$.*

Observe that Theorem 1 is an immediate consequence of Propositions 2 and 12.

Proof. Suppose, first of all, that N is a normal subgroup of G that contains H. Since $G = H.K$ we conclude that $K \not\leq N$ and thus that $K \cap N = \{1\}$. Since H is a subgroup of N we conclude, in addition that $G = N.K$. If H is a proper subgroup of N, then $H.K$ is a proper subset of G, which is a contradiction. Thus $H = N$, i.e. H is normal in G. Then, since $G = H.K$ and $H \cap K = \{1\}$ we conclude that $G = H \rtimes K$. But now, by considering $F^*(G)$, the generalized Fitting subgroup of G, we conclude immediately that $G = H \times K$ as required.

Suppose, on the other hand, that neither H nor K are subgroups of any proper normal subgroup of G. If G is simple, then the result follows from Proposition 2. Assume, then that N is a maximal proper non-trivial subgroup of G and we will demonstrate a contradiction. Observe, first, that $G/N \cong S$, a non-abelian simple group. Furthermore $G/N = (HN/N).(KN/N)$.

Suppose that N_1 is a maximal proper non-trivial subgroup of G that is distinct from N. Then the same argument implies that $G/N_1 \cong S_1$, another non-abelian simple group. Then $G/N \cap N_1 \cong S \times S_1$. By considering cardinalities, one quickly concludes that (without loss of generality) $H \cong S$ and $K \cong S_1$ and $N = \{1\}$. Thus the result holds in this case.

Assume, then, that N is the only maximal proper non-trivial subgroup of G. Note, moreover, that since $|K|$ divides $|G/N|$, $|N|$ divides $|H|$. Let M be a second-maximal normal subgroup of G, i.e. M is a normal subgroup of G and N is the only proper normal subgroup of G that properly contains M. Then $N/M \cong \underbrace{T \times \cdots \times T}_{d}$ where d is a finite integer and T is a finite simple group. Since N/M is the only proper non-trivial subgroup of G/M, it is clear that $F^*(G/M) = N/M$. In particular, the outer automorphism group of T^d, which is isomorphic to $\mathrm{Out}(T) \wr \mathrm{Sym}(d)$ contains a subgroup isomorphic to $G/N \cong S$.

Now we appeal to the Schreier conjecture to conclude that K is not a subgroup of the outer automorphism group of T and, therefore, S is isomorphic to a subgroup of $\mathrm{Sym}(d)$. Let p be a prime dividing $|T|$. Then p^d divides $|N|$ and so divides $|H|$, which in turn divides $|S|$ which in turn divides $d!$. But this is a contradiction of Lemma 3 and we are done. □

References

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