On Cherlin's Conjecture

Nick Gill (USW)

20th January 2016

Joint with
Hunt (USW) and Spiga (Milano–Bicocca).

Relational structures

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Definition

A **relational structure** $\mathcal{S}$ *is a tuple* $(\Omega, R_1, R_2, \ldots, R_k)$ *where*

- Ω *is a (finite) set;*
- *For all* $i = 1, \ldots, k$*, there is an integer* ℓ_i *such that*

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The sets $R_1, \dots, R_k$ are called **relations**. The relation R_1 is an ℓ_1**-ary** relation. If $\ell_1 = 2$, then we say that R_1 is a **binary** relation.

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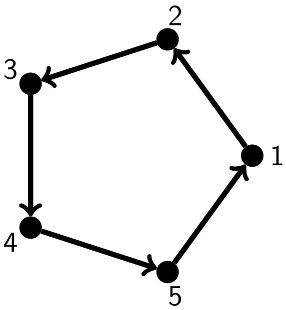

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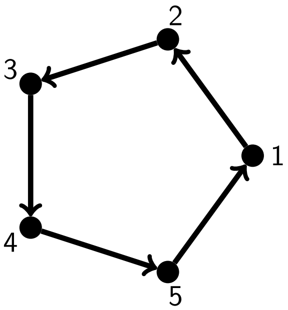

The above directed graph is a representation of the relational structure

$$(\{1,2,3,4,5\},\{(1,2),(2,3),(3,4),(4,5),(5,1)\})\,.$$

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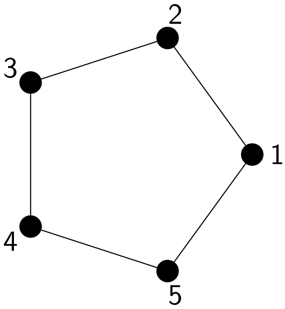

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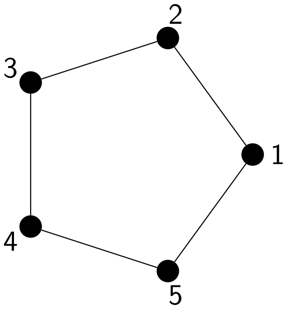

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This notion of an automorphism just extends the accepted definition of an automorphism of a (directed) graph.

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A relational structure $\mathcal{S}$ is called **homogeneous** *if, given two induced substructures $\mathcal{S}_1$ and $\mathcal{S}_2$ and an isomorphism $\psi : \mathcal{S}_1 \to \mathcal{S}_2$, there is an automorphism $\phi \in \mathrm{Aut}(\mathcal{S})$ such that $\phi|_{\mathcal{S}_1} = \psi$.*

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In other words, every local symmetry in the relational structure extends to a global symmetry of the overall structure.

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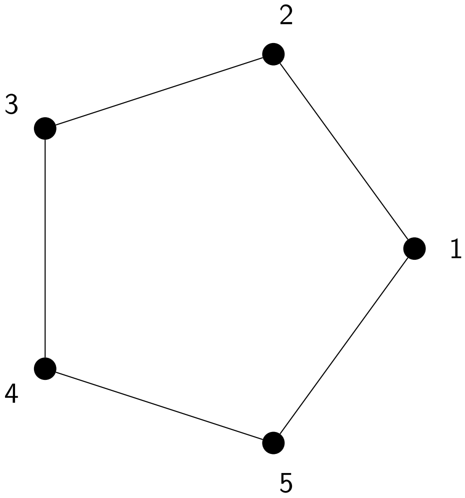

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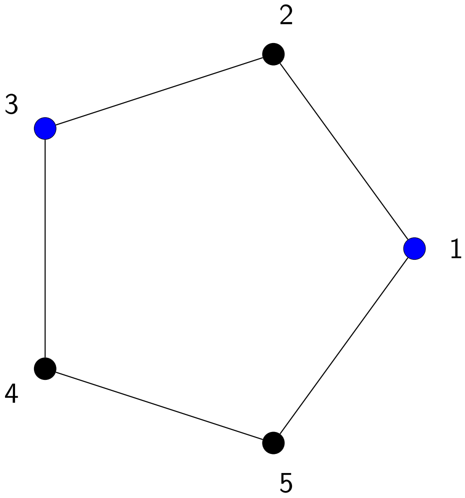

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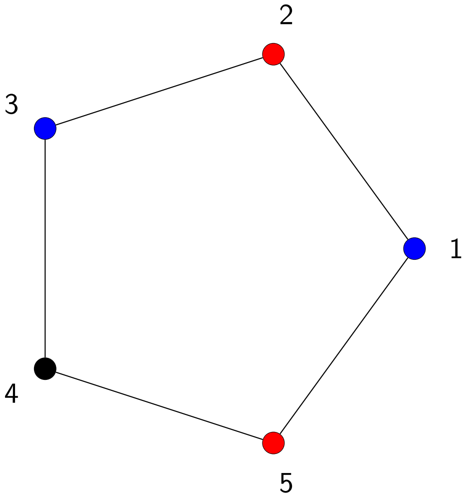

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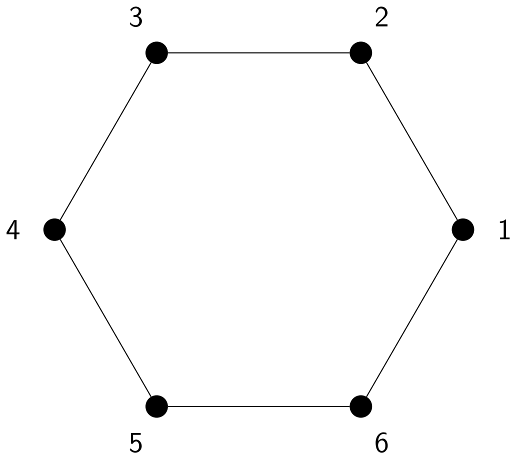

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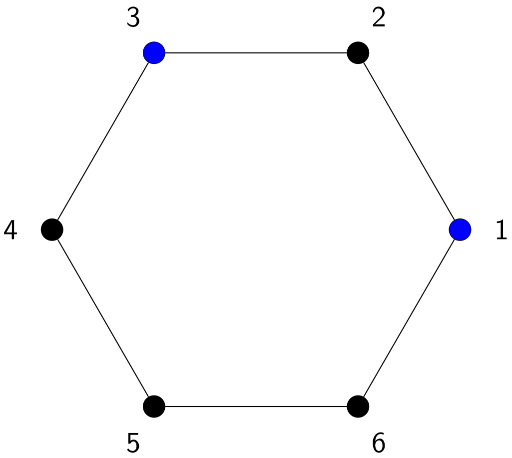

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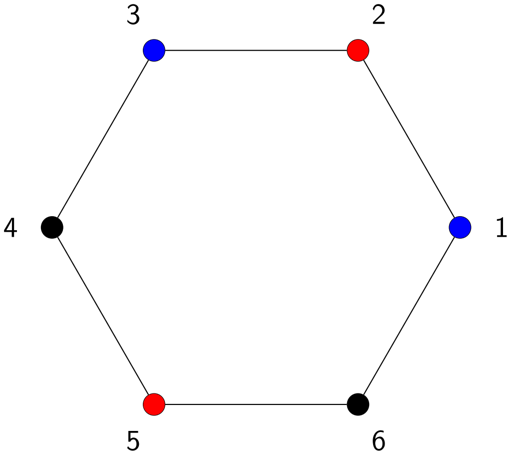

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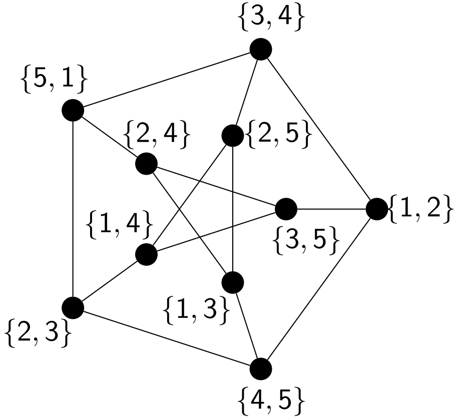

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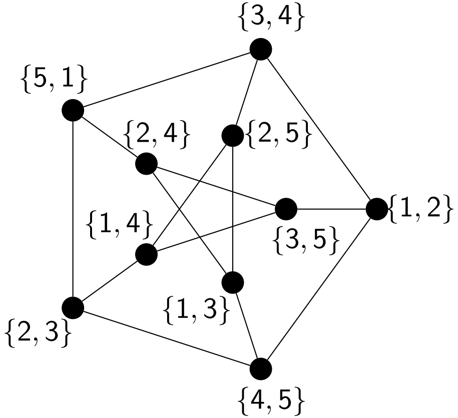

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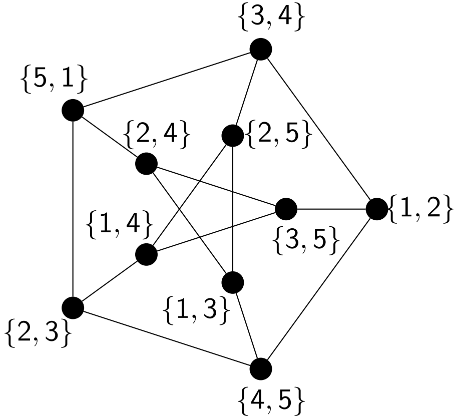

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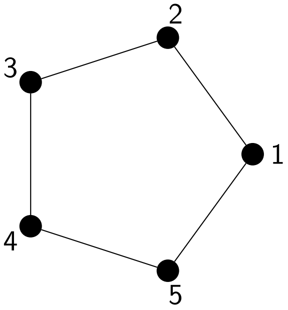

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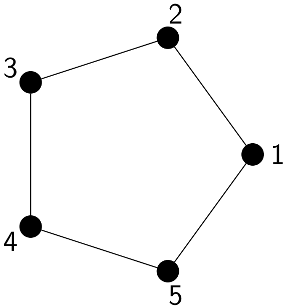

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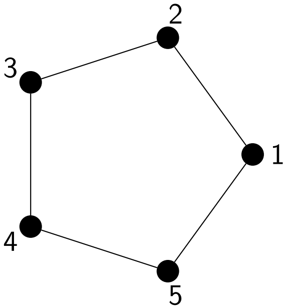

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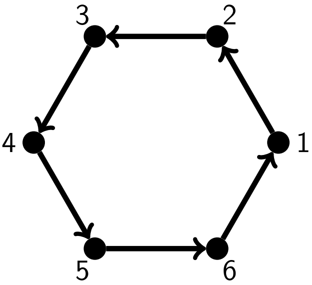

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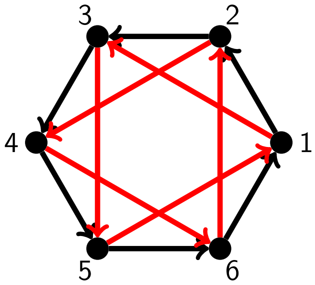

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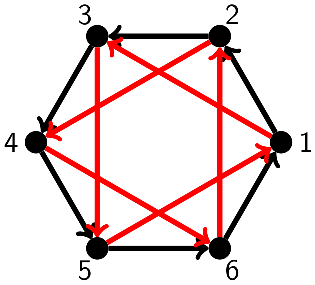

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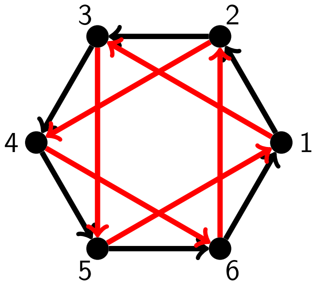

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3. G *is an affine orthogonal group* $V \cdot O(V)$, *and* $\Omega = V$.

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4. We conclude that $G = \mathrm{Aut}(\mathcal{S}) = \mathrm{Sym}(\Omega)$.

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Our method is to study the almost simple primitive actions and show that they (nearly) always contain a barely 2-transitive subset.

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4. This fact (virtually) yields Cherlin's conjecture for the alternating and symmetric groups.

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    1. $G = \mathrm{Sym}(\Omega)$, or
    2. $n = p$, a prime, and $M \cong C_p \rtimes C_{\frac{p-1}{2}}$.
4. This fact (virtually) yields Cherlin's conjecture for the alternating and symmetric groups.
5. We are left with the classical groups, exceptional groups and sporadic groups...

Thanks for coming!