On Cherlin's Conjecture

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Overview

Beautiful sets

From here on, G is a group acting on a set Ω, and Λ is a subset of Ω.

- Write G_Λ for the *set-wise stabilizer* of Λ.
- Write $G_{(\Lambda)}$ for the *point-wise stabilizer* of Λ.
- Write $G^\Lambda = G_\Lambda / G_{(\Lambda)}$ for the *induced permutation group* on Λ.

Definition

We say that Λ is a **beautiful set** *if G^Λ acts 2-transitively on Λ but G^Λ does not contain $\mathrm{Alt}(\Lambda)$.*

What price beauty?

It is our contention that, in the universe of primitive permutation groups, beautiful sets crop up very often. How will we find them?

- If Λ is beautiful, then $|\Lambda| \geq 5$.
- Suppose that G is almost simple with socle S. If Λ is beautiful with respect to S, then Λ is beautiful with respect to G.
- Thus, to find beautiful sets for almost simple groups, it is enough to find beautiful sets for simple groups, where the action might no longer be primitive.

A proposition about classical groups

Proposition

Let S be a finite simple classical group. Let M be a subgroup of S from Aschbacher's class $\mathcal{C}_1$, and let Ω be the set of cosets of M in S. There are two possibilities:

1. *There is a beautiful set.*
2. *(S, M) is on a finite list of known exceptions.*

Example: for $S = \mathrm{PSL}_n(q)$, the exceptions are:

1. $S = \mathrm{SL}_2(4)$;
2. $S = \mathrm{SL}_3(2)$, $M = \mathrm{Stab}(W_1, W_2)$, $V = W_1 \oplus W_2$, $\dim(W_1) = 1$, $\dim(W_2) = 2$;
3. $S = \mathrm{SL}_3(2), \mathrm{SL}_3(3)$, $M = \mathrm{Stab}(W_1, W_2)$, $W_1 \subset W_2$, $\dim(W_1) = 1$, $\dim(W_2) = 2$.

Strategy of proof

- We look for Frobenius groups $[r^a] : (r^a - 1)$ **embedded in** S **in the right way**.
- Let's suppose that $S = \mathrm{SL}_n(q)$ and M is maximal parabolic.
- Elements of S look like:

$$\begin{pmatrix} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}$$

$$\begin{pmatrix} 1 & * & * & 0 & 0 & 0 \\ * & 1 & * & 0 & 0 & 0 \\ * & * & a & 0 & 0 & 0 \\ * & * & * & 1 & * & * \\ * & * & * & * & 1 & * \end{pmatrix} \quad C_{p-1}$$

Extensions

- We are working on versions of this proposition for the other geometric subgroups.
- These other version aren't all as strong. For example:
  - let $S = \mathrm{PSL}_2(p)$ with p odd;
  - let $M \in \mathcal{C}_2$, i.e. $M \cong D_{p-1}$ is the normalizer of a split torus;
  - there is no beautiful set for this action.
- Most commonly, beautiful sets become scarce if either the dimension or the field are small.
- There are also versions of this proposition for various actions of the alternating groups.
- Exceptionals? Sporadics?

Relational structures

Definition

A **relational structure** $\mathcal{S}$ *is a tuple* $(\Omega, R_1, R_2, \dots, R_k)$ *where*

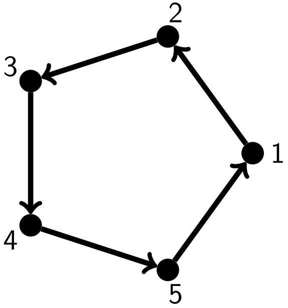
- Ω *is a (finite) set;*
- *For all* $i = 1, \dots, k$*, there is an integer* ℓ_i *such that*

$$R_i \subseteq \underbrace{\Omega \times \Omega \times \cdots \times \Omega}_{\ell_i}.$$

The sets $R_1, \dots, R_k$ are called **relations**. The relation R_1 is an ℓ_1**-ary** relation. If $\ell_1 = 2$, then we say that R_1 is a **binary** relation.

An example of a relational structure

You should think of relational structures as a generalization of simple, directed graphs.



The above directed graph is a representation of the relational structure

$$(\{1,2,3,4,5\},\{(1,2),(2,3),(3,4),(4,5),(5,1)\}).$$

Automorphisms

Definition

An **automorphism** *of a relational structure* $(\Omega, R_1, \ldots, R_k)$ *is a permutation* $\phi \in \mathrm{Sym}(\Omega)$ *such that*

$$(\omega_1, \ldots, \omega_{\ell_i}) \in R_i \textit{ for some } i \Longrightarrow (\phi(\omega_1), \ldots, \phi(\omega_{\ell_i})) \in R_i.$$

This notion of an automorphism just extends the accepted definition of an automorphism of a (directed) graph.

Homogeneity

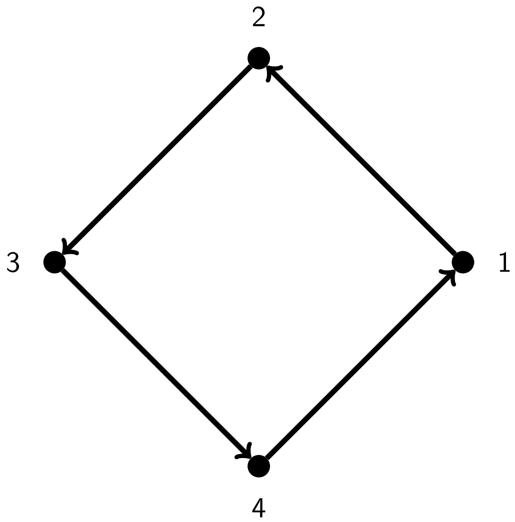
"Local symmetry implies global symmetry".

Definition

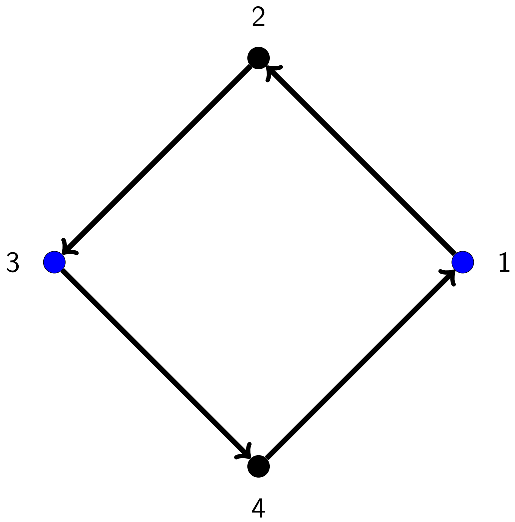
A relational structure $\mathcal{S}$ is called **homogeneous** *if, given two induced substructures $\mathcal{S}_1$ and $\mathcal{S}_2$ and an isomorphism $\psi : \mathcal{S}_1 \to \mathcal{S}_2$, there is an automorphism $\phi \in \mathrm{Aut}(\mathcal{S})$ such that $\phi|_{\mathcal{S}_1} = \psi$.*

In other words, every local symmetry in the relational structure extends to a global symmetry of the overall structure.

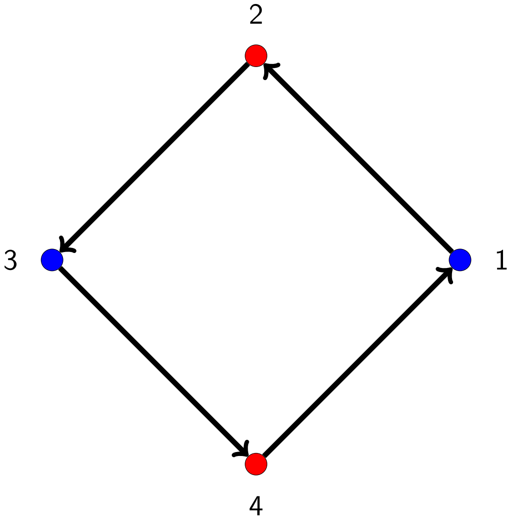
A homogeneous directed graph



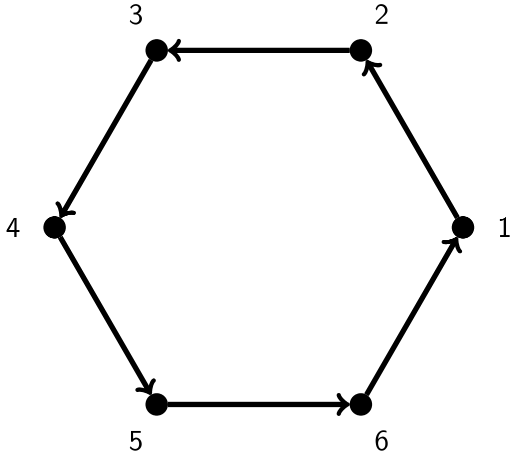
A homogeneous directed graph



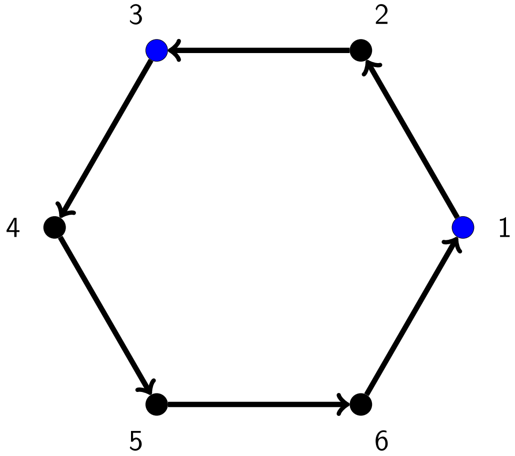
A homogeneous directed graph



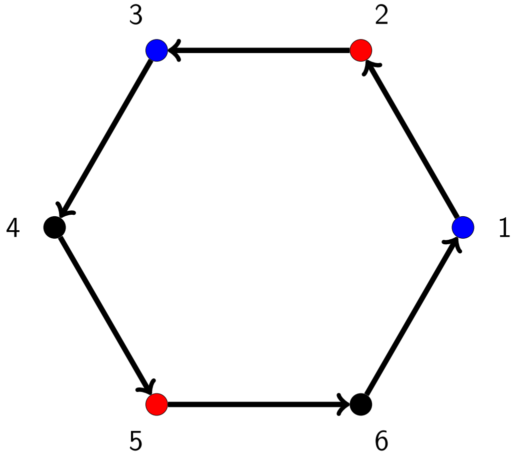
A nonhomogeneous directed graph



A nonhomogeneous directed graph



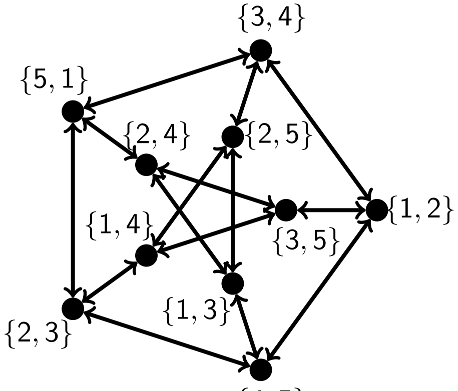
A nonhomogeneous directed graph



Example: adding structures to group actions I

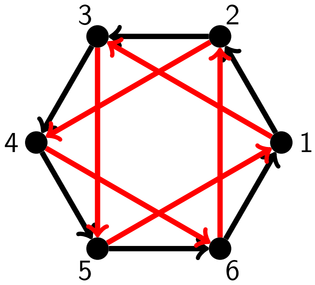
Consider the group $G = \mathrm{Sym}(5)$ acting naturally on the set Ω of distinct 2-subsets of the set $\{1, 2, 3, 4, 5\}$.

$$\Omega = \left\{ \begin{array}{c} \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\} \\ \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\} \end{array} \right\}$$



Example: adding structures to group actions II

Consider the cyclic group $G = C_6 = \langle(1,2,3,4,5,6)\rangle$ acting naturally on the set $\Omega = \{1,2,3,4,5,6\}$.



Let $\mathcal{S} = (\Omega, R_1, R_2)$. Then $G = \mathrm{Aut}(\mathcal{S})$ and $\mathcal{S}$ is homogeneous.

Binary actions

1. Suppose that a group G acts on a set Ω.
2. Suppose that $\mathcal{S} = (\Omega, R_1, \ldots, R_k)$ is a relational structure on Ω such that $G = \mathrm{Aut}(\mathcal{S})$. We say that $\mathcal{S}$ is **compatible** with the action.
3. We call the action **binary** if there is a compatible *homogeneous* relational structure for which all of the relations are binary.

Some examples:

1. C_n acting on $\{1, 2, \ldots, n\}$ is binary.
2. $\mathrm{Sym}(5)$ acting on the set of distinct pairs is not binary (it has relational complexity equal to 3).
3. $\mathrm{Sym}(n)$ acting on $\{1, 2, \ldots, n\}$ is binary.

Cherlin's Conjecture

Conjecture (Cherlin)

Suppose that a finite group G acts faithfully and primitively on a set Ω. If the action is binary, then it is "known".

By work of Wiscons, we need only consider the situation when G is (almost) simple.

Conjecture

Suppose that a finite almost simple group G acts faithfully and primitively on a set Ω. If the action is binary, then $G = \mathrm{Sym}(\Omega)$.

Towards a proof

Lemma

If the action of G on Ω is 2-transitive and binary, then $G = \mathrm{Sym}(\Omega)$.

Proof.

1. Let $\mathcal{S} = (\Omega, R_1, \ldots, R_k)$ be a homogeneous structure that is compatible with the action and for which $R_1, \ldots, R_k$ are all binary.
2. If $(\omega_1, \omega_2) \in R_i$, then $(\omega_1, \omega_2)^g \in R_i$ for all $i = 1, \ldots, k$.
3. 2-transitivity $\Longrightarrow$ R_i is equal to $\Omega^{(1)}$ or $\Omega^{(2)}$ or Ω^2.
4. We conclude that $G = \mathrm{Aut}(\mathcal{S}) = \mathrm{Sym}(\Omega)$.

□

Beautiful sets

We can extend the previous lemma to get the following.

Lemma

Suppose that the action of G on Ω is binary. If Λ is a subset of Ω for which G^Λ is 2-transitive, then $G^\Lambda = \mathrm{Sym}(\Lambda)$. In particular, Ω does not contain a beautiful subset.

Thus, for all of the actions where we have found a beautiful subset, Cherlin's conjecture holds.

Thanks for coming!