MATRICES FOR $\mathrm{O}_8^+(q)$, $G_2(q)$ AND ${}^3D_4(q)$

NICK GILL

1 August 2024: Daniele Garzoni pointed out a couple of things that needed correcting, plus some extra observations, so I have edited accordingly. Grazie mille, Daniele!

1. Introduction

I want to try and understand the triality automorphism for $\mathrm{O}_8^+(q)$ from the "classical" point of view. The presence of triality makes sense if we think of $\mathrm{O}_8^+(q)$ as $D_4(q)$, but I fail to see how triality is natural when we consider the usual 8-dimensional vector space over $\mathbb{F}_q$. My main text for reference is [Car89].

My investigations start by listing the positive D_4 roots, assigning a roman letter to each one.:

Height	Roots			
1	$(a)\ \substack{100\\0}$	$(b)\ \substack{010\\0}$	$(c)\ \substack{001\\0}$	$(d)\ \substack{000\\1}$
2	$(e)\ \substack{110\\0}$	$(f)\ \substack{011\\0}$	$(g)\ \substack{010\\1}$	
3	$(h)\ \substack{110\\1}$	$(i)\ \substack{011\\1}$	$(j)\ \substack{111\\0}$	
4	$(k)\ \substack{111\\1}$			
5	$(\ell)\ \substack{121\\1}$			

Table 1.1. Roots of D_4

Now I can use the assigned Roman letter to identify the root groups in, U, a Sylow p-subgroup of $\mathrm{O}_8^+(q)$. Consider an ordered hyperbolic basis:

$$\mathcal{B} = \{e_1, e_2, e_3, e_4, f_4, f_3, f_2, f_1\}.$$

Now we can think of root groups using the following diagram:

$$\begin{pmatrix} 1 & a & e & j & h & k & \ell & \\ & 1 & b & f & g & i & & -\ell \\ & & 1 & c & d & & -i & -k \\ & & & 1 & & -d & -g & -h \\ & & & & 1 & -c & -f & -j \\ & & & & & 1 & -b & -e \\ & & & & & & 1 & -a \\ & & & & & & & 1 \end{pmatrix}.$$

This diagram should be interpreted as follows. The root group X_a is the one corresponding to the root a in the table above. It is obtained by setting all other letters in the matrix to 0, and allowing the variable a to range over all values of $\mathbb{F}_q$. The q matrices so obtained form an elementary-abelian group of order q. The same applies for all other roots.

In this way one obtains 12 root groups and now U is the product of these. Note that you *don't* get the elements of U by just letting the 12 variables in the matrix above range over $\mathbb{F}_q$.

To get a look at triality, one might try to write down the four parabolic subgroups containing P. Let's label these as P_a, P_b, P_c and P_d. To obtain P_a we take Q and then throw in all negative fundamental root groups apart from X_{-a} (the negative root group being just the transpose of the positive one).

Of the four resulting parabolics, we find that P_b is of one isomorphism class (it stabilizes a 2-space), while the others are all isomorphic (joined by triality) with P_a stabilizing a 1-space, and P_c and P_d stabilizing different 4-spaces.

The Levi-factors of P_a, P_c and P_d are all of type A_3 – let us see how they intersects Q in each case (using a similar scheme to above):

$$Q\cap P_a:\begin{pmatrix}1&&&&&&&\\&1&b&f&g&i&&\\&&1&c&d&&-i&\\&&&1&&-d&-g&\\&&&&1&-c&-f&\\&&&&&1&-b&\\&&&&&&1&\\&&&&&&&1\end{pmatrix},\quad Q\cap P_c:\begin{pmatrix}1&a&e&&h&&&\\&1&b&&g&&&\\&&1&&d&&&\\&&&1&&-d&-g&-h\\&&&&1&&&\\&&&&&1&-b&-e\\&&&&&&1&-a\\&&&&&&&1\end{pmatrix}$$

$$\text{and } Q\cap P_d:\begin{pmatrix}1&a&e&j&&&&\\&1&b&f&&&&\\&&1&c&&&&\\&&&1&&&&\\&&&&1&-c&-f&-j\\&&&&&1&-b&-e\\&&&&&&1&-a\\&&&&&&&1\end{pmatrix}.$$

Notice that $Q\cap P_a$ "looks like" the Sylow p-subgroup of $\mathrm{O}_6^+(q)$ which is of type $D_3=A_3$. On the other hand $Q\cap P_d$ is very clearly isomorphic to the sylow P-subgroup of $\mathrm{SL}_4(q)$ which is of type A_3. And finally, $Q\cap P_c$ looks like, well, a corrupted version of $\mathrm{SL}_4(q)$. Not so insightful then – I still don't feel like I really *get* triality when I think about the classical theory of $\mathrm{O}_8^+(q)$. I await inspiration to strike...

2. $G_2(q)$

An unexpected bonus coming out of this analysis was that I can write down the matrices for the "natural" 8-dimensional representation of $G_2(q)$. This is because $G_2(q)$ can be seen as the centralizer of a triality automorphism g of $\mathrm{O}_8^+(q)$. In terms of roots, we have:

G_2 root	Length	Corresponding $\mathrm{O}_8^+(q)$ roots		
(A) 01	short	(a) $\substack{100\\0}$	(c) $\substack{001\\0}$	(d) $\substack{000\\1}$
(B) 10	long	(b) $\substack{010\\0}$		
(C) 11	short	(e) $\substack{110\\0}$	(f) $\substack{011\\0}$	(g) $\substack{010\\1}$
(D) 12	short	(h) $\substack{110\\1}$	(i) $\substack{011\\1}$	(j) $\substack{111\\0}$
(E) 13	long	(k) $\substack{111\\1}$		
(F) 23	long	(ℓ) $\substack{121\\1}$		

TABLE 2.1. Roots of G_2

As before I have assigned a Roman letter to each root of G_2, this time the letter being capital. The corresponding root group in G_2 will then be a subgroup of a product of some root groups in $O_8^+(q)$. When the root is long, the root groups of $G_2(q)$ correspond to root groups of $O_8^+(q)$. If the root is short, we do like this:

$$X_A:=\{x_a(t).x_c(t).x_d(t)\mid t\in\mathbb{F}_q\},$$

so X_A is some kind of "diagonal" subgroup of the product $X_aX_cX_d$ (note that, thanks to living in a beneficent universe, this product is itself a group). One needs to be a bit careful writing down the actual elements of X_A. They are:

$$\begin{pmatrix}1&t&&&&&&\\&1&&&&&&\\&&1&t&t&-t^2&&\\&&&1&&-t&&\\&&&&1&-t&&\\&&&&&1&&\\&&&&&&1&-t\\&&&&&&&1\end{pmatrix},$$

with t ranging over $\mathbb{F}_q$. Nonetheless, if one is willing to take care in this way, then one can represent the root groups of $G_2(q)$ using a similar scheme to that previously, modifying the interpretation of short root groups appropriately.

$$
(1)\qquad \begin{pmatrix}
1 & A & C & D & D & E & F & -F^2 \\
 & 1 & B & C & C & D & -D^2 & -F \\
 & & 1 & A & A & -A^2 & -D & -E \\
 & & & 1 & & -A & -C & -D \\
 & & & & 1 & -A & -C & -D \\
 & & & & & 1 & -B & -C \\
 & & & & & & 1 & -A \\
 & & & & & & & 1
\end{pmatrix}.
$$

I didn't bother specifying the split torus above for $O_8^+(q)$ as it is obvious. However, for the $G_2(q)$ case, it is worth noting it down (using Carter's results about the H-group in [Car89]). It is the product of these two groups:

$$
\begin{aligned}
H_A &= \mathrm{diag}[y,\, y^{-1},\, y^2,\, 1,\, 1,\, y^{-2},\, y,\, y^{-1}] \\
H_B &= \mathrm{diag}[1,\, x,\, x^{-1},\, 1,\, 1,\, x,\, x^{-1},\, 1]
\end{aligned}
$$

where x and y, as usual, range over $\mathbb{F}_q$.

Working out the parabolic subgroups for $G_2(q)$ is nice and easy – in both cases the Levi factor is a $\mathrm{GL}_2(q)$ and the unipotent radical has size q^5. I find it interesting, though, to see that the action of the Levi on the unipotent radical is different in each:

(1) In P_B, where we add the root X_{-A} to our positive root groups, we find that the Levi factor normalizes X_F.
(2) In P_A, where we add the root X_{-B} to our positive root groups, we find that the Levi factor does not normalize any single root group in the unipotent radical. Instead it normalizes $X_E.X_F$.

2.1. **A remark from Daniele.** Although the description just given is about the embedding $G_2(q) \leq O_8^+(2)$, one can observe that in fact the subgroup $G_2(q)$ stabilizes the 7-subspace $W = \langle e_1, e_1, e_3, _4 + f_4, f_3, f_2, f_1\rangle$, centralizing $e_4 - f_4$. So we can easily obtain an embedding $G_2(q) < O_7(q)$.

Further, if q is even, then the subgroup $G_2(q)$ centralizes $e_4 + f_4$ so if we pass to the action on the quotient $W/\langle e_4 + f_4\rangle$ (for which we can use the basis $\{e_1, e_2, e_3, f_3, f_2, f_1\}$), then we get a concrete description of the embedding into $\mathrm{Sp}_6(q)$.

3. ${}^3D_4(q)$

The next bit really surprised me: pretty much all the work we did for $G_2(q)$ now carries over to $G = {}^3 D_4(q)$, by remembering that we can see G as the centralizer in $O_8^+(q^3)$ of gf where g is the triality automorphism discussed above, and $f : t \mapsto t^q$ is the Frobenius automorphism.

Now the roots for G_2 listed in Table 2.1 also apply for ${}^3D_4(q)$ with the same root groups of $O_8^+(q)$ here as there. Similarly the scheme written at (1) also applies, however, we should just be careful to adjust our interpretation:

(1) If the corresponding G_2 root is long, then our root group is obtained by taking the corresponding root group in $O_8^+(q^3)$ and ranging over $\mathbb{F}_q$. So, for example,
$$X_B := \{x_b(t) \mid t \in Fq\}.$$
(2) If the corresponding G_2 root is short, then we need, first, to be careful with our choice of g. Let's take g here to be the automorphism that moves the roots of D_4 as follows
$$g : (a) \longrightarrow (c) \longrightarrow (d) \longrightarrow (a).$$
Then we must adjust the root as we go along by f. Thus,
$$X_A := \{x_a(t).x_c(t^q).x_d(t^{q^2}) \mid t \in \mathbb{F}_{q^3}\}.$$
It is important to observe that changing the order of the product in the short roots doesn't make any difference, i.e.
$$x_a(t_1)x_c(t_2)x_d(t_3) = x_a(t_1)x_d(t_3)x_c(t_2) = x_d(t_3)x_a(t_1)x_c(t_2) = \cdots$$

and this guarantees that X_A really does centralize gf. It's probably worth just noting down what the elements of X_A look like as matrices:

$$\begin{pmatrix} 1 & t & & & & & & \\ & 1 & & & & & & \\ & & 1 & t^q & t^{q^2} & -t^{q+q^2} & & \\ & & & 1 & & -t^{q^2} & & \\ & & & & 1 & -t^q & & \\ & & & & & 1 & & \\ & & & & & & 1 & -t \\ & & & & & & & 1 \end{pmatrix},$$

As before we can write down the split torus here. It is the product of these two groups:

$$H_A = \mathrm{diag}[y,\, y^{-1},\, y^{q^2+q},\, y^{q^2-q},\, y^{q-q^2},\, y^{-q-q^2},\, y,\, y^{-1}]$$
$$H_B = \mathrm{diag}[1,\, x,\, x^{-1},\, 1,\, 1,\, x,\, x^{-1},\, 1]$$

where x ranges over $\mathbb{F}_q$ and y ranges over $\mathbb{F}_{q^3}$.

Again, working out the parabolics is easy.

(1) In P_B, where we add the root X_{-A} to our positive root groups, we find that the Levi factor is of type $A_1(q^3)$.
(2) In P_A, where we add the root X_{-B} to our positive root groups, we find that the Levi factor is of type $A_1(q)$.

Analysing the action on the unipotent radical is straightforward.

References

[Car89] Roger W. Carter. *Simple groups of Lie type. Reprint of the 1972 orig.* New York: John Wiley &— Sons, Inc., reprint of the 1972 orig. edition, 1989.

Department of Mathematics, University of South Wales, Treforest, CF37 1DL, U.K.
Email address: nick.gill@southwales.ac.uk